Definability in functional analysis

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Abstract. The role played by real-valued functions in functional analysis is fundamental. One often considers metrics, or seminorms, or linear functionals, to mention some important examples.

We introduce the notion of definable real-valued function in functional analysis: a real-valued function f defined on a structure of functional analysis is *definable* if it can be "approximated" by formulas which do not involve f. We characterize definability of real-valued functions in terms of a purely topological condition which does not involve logic.

1. Introduction

In this paper we consider the following question. Let E be a structure from functional analysis (e.g., a Banach space) and let f be a real-valued function defined on E (e.g., the norm, or a linear functional, or the distance to a subspace): when is f definable? Can the condition "f is definable" be characterized in terms which do not involve logic? We claim to answer this question affirmatively for a rich class of structures.

If E is a structure from functional analysis, one generally considers two types of functions on E. (1) *Operations*, i.e., functions of the form $f : E^n \to E$, where n is a positive integer; (2) *Real-valued functions*, i.e., functions of the form $\mathcal{R} : E^n \to [-\infty, \infty]$, where n is a positive integer and $[-\infty, \infty]$ denotes the extended real numbers. We shall consider structures of the form

$$\mathbf{E} = (\, E, f_i, \mathcal{R}_j \,)_{i \in I, j \in J},$$

where E (the *universe* of $\mathbf{E}$) is a metric space, each f_i is an operation, and each $\mathcal{R}_j$ is a real-valued function. We will require some further conditions on the universe and the functions, namely: (1) E is a metric space; (2) Each $\mathcal{R}_j$ is uniformly continuous on the bounded subsets E; (3) Each f_i is uniformly continuous and bounded on the bounded subsets of E. We shall refer to these structures as *metric space structures*. There are two important reasons to restrict our attention to this class of structures:

- This is precisely the class of structures for which *ultrapowers*, in the sense considered in functional analysis, are defined;
- They have a rich model theory.

The notion of *ultrapower* in functional analysis was introduced in [2], motivated by the ultrapower construction from model theory. (For a survey, see [4].) The construction has now become standard in other areas of functional analysis. Some of the most celebrated results of Banach space theory in the last two decades have been proved with the use of ultrapowers. (See, for example, [3]; also [1] and [9].)

First-order logic is not appropriate for functional analysis. (This statement was first made precise in [8].) A functional-analytic ultrapower of a metric space structure $\mathbf{E}$ is not an elementary extension of $\mathbf{E}$. However, this is true in a logic weaker than first-order logic introduced by C. W. Henson in [6] and [7]. (See also [5]).

A model theoretical framework for the study of metric space structures was formally introduced in [8]. In this context, functional-analytic ultrapowers play a role analogous to that played by algebraic ultrapowers in first-order model theory.

In the model theory of metric space structures, as well as in first-order model theory, one deals with the abstract notion of elementary extension rather than with concrete ultrapowers. However, the advantage of ultrapowers lies two facts: (1) Their construction involves only notions from analysis (as opposed to logic); (2) Their usefulness is now widely recognized in functional analysis. Thus, when a model theoretical notion about metric space structures lends itself to be rephrased in terms of elementary extensions (e.g., elementary equivalence), results involving this notion can be made readily accessible to the analysis community by replacing elementary extensions with ultrapowers.

Here we study the notion of definability in functional analysis. Two differences between the notion of definablity in analysis and the usual, rather algebraic one from first-order model theory must be emphasized. First, in analysis, one should study definable *real-valued functions* rather than definable *sets*. It became clear in [8] that the role played by the former in functional analysis mirrors that played by the latter in first-order model theory. (To illustrate this with an example, let E be a metric space and let E_0 be a subset of E. In algebra, one would consider the boolean-valued relation "$x \in E_0$"; however, in analysis, one generally needs more information, namely, the real-valued function "distance to the set E_0".) Thus, in order to provide a nonlogical characterization of definability, one must characterize definability of real-valued functions, rather than definability of sets.

Secondly, the notion of definability in functional analysis is somewhat more involved than its first-order counterpart. In the latter field, a set is definable if it is completely determined by a single first-order formula; in the former, a real-valued function if definable if its values can be can be "uniformly approximated" at will by positive bounded formulas. The subtleties arise in that the set of parameters needed will depend in general on the accuracy of the approximation.

Let $\mathbf{E}$ be a metric space structure and let $\mathcal{R}$ be a real-valued function on E^n, where E is the universe of $\mathbf{E}$. We want to characterize the property "there exists a family of formulas $(\phi_\epsilon)_{\epsilon>0}$ which uniformly approximates $\mathcal{R}$". If all the formulas ϕ_ϵ are over a given subset A of E, the answer is given by a functional-analytic version of Svenonious' theorem (Corollary 4). However, this answer is not entirely satisfactory because the set of parameters need not be known beforehand; in fact, in general practice, the parameters depend on the accuracy of the approximation, as well as the specific nature of the structure. Thus, the question should be restated as follows: Let $\mathcal{R}$ be a real-valued function on E; for a subset A of E, when is it true that every model containing A will contain enough parameters to approximate $\mathcal{R}$ as closely as desired? In other words, when is it true that whenever $(\mathbf{E}', \mathcal{R}', a)_{a\in A}$ is elementary equivalent to $(\mathbf{E}, \mathcal{R}, a)_{a\in A}$, the function $\mathcal{R}'$ is definable? Corollary 6 provides an answer in terms of a topological condition satisfied by sufficiently rich ultrapowers of $(\mathbf{E}, \mathcal{R})$.

In classical model theory, the above condition corresponds to Shelah's notion of being *almost over* A, and is characterized in terms of definable equivalence relations on the models of the theory. In the funtional analytic context, this role is played by definable *pseudometrics* on bounded subsets of the models of the theory.

In Section 1, we introduce the notion of definability for metric space structures, and study the topological properties of these relations which will be used in the rest of the paper. Section 2 and 3 are devoted to the main question ("when is a real-valued function $\mathcal{R}$ definable?"). In Section 2, we answer the question for the particular case when the set parameters of the formulas which approximate $\mathcal{R}$ are known. As we have mentioned above, this set of parameters is in general not given. Nonetheless, the result of Section 2 (Theorem 2) is used both as a lemma and as a natural motivation for the general result (Theorem 5) in Section 3, which is the central section of the paper. Finally, in Section 4, we apply the results from Sections 2 and 3 to study the question: when is a type definable?. The notion of type has already proved quite useful in Banach space geometry. See, for example, [11], [12], [13], and [14].

The shall assume that the reader is familiar with the basic machinery for the model theoretical analysis of metric space structures introduced in [8]. As in [10], we shall restrict our attention *normed space structures* (i.e., structures whose universe is a normed space) both in order to focus on mathematically richer structures and to simplify the exposition. Nonetheless, the results can be easily modified for the general context of metric space structures.

2. Definable real-valued relations

We work under the notational conventions of [10]. Namely, T denotes the complete positive bounded theory of a normed space structure over a countable language L. All the models considered are models of T. We rule out the trivial case when T is the complete theory of the Banach space $\{0\}$.

We also assume that all the models of T are approximately elementary submodels of some large, saturated model $\mathfrak{C}$. By the saturation of $\mathfrak{C}$, we have $\mathfrak{C} \models_{\mathcal{A}} \phi(\bar{a})$ if and only if $\mathfrak{C} \models \phi(\bar{a})$ for every positive bounded formula ϕ and every $\bar{a} \in \mathfrak{C}$; We write simply $\models \phi(\bar{a})$, omitting $\mathfrak{C}$. If $\bar{a} \in \mathfrak{C}$, we write $\|\bar{a}\|$ as an abbreviation of $\max_{1 \leq i \leq n} \|a_i\|$.

We consider only complete, positive bounded types which are consistent with T. The norm of an n-type p, denoted $\|p\|$, is the norm of any n-tuple realizing the type. If X is a subset of $\mathfrak{C}$ or a set of types, we denote by $\mathscr{B}_N(X)$ the set of elements of X of norm $\mathscr{N}$; if $N = 1$, we write $\mathscr{B}(X)$ instead of $\mathscr{B}_1(X)$; if $X = \mathfrak{C}$, we write $\mathscr{B}_N$ (or $\mathscr{B}$) instead of $\mathscr{B}_N(\mathfrak{C})$ (or $\mathscr{B}(\mathfrak{C})$).

As in [8], if $\mathbf{E} = (\, E, f_i, \mathcal{R}_j \,)_{i \in I, j \in J}$, we will call the $\mathcal{R}_j$'s *real-valued relations*, instead of real-valued functions. In order to simplify the exposition, we will assume that all the real-valued relations take values on the closed unit interval $[0, 1]$. This is a rather harmless assumption, since the interval $[-\infty, \infty]$ is uniformly homeomorphic to $[0, 1]$.

Let $N > 0$. An m-ary *real-valued relation on $\mathscr{B}_N$* is a uniformly continuous function $\mathcal{T} : (\mathscr{B}_N)^m \to [0, 1]$. Thus, $\mathcal{T}$ is a real-valued relation on $\mathscr{B}_N$ if and only if $\mathcal{T} = \mathcal{R} | \mathscr{B}_N$, where $\mathcal{R}$ is a real-valued relation on $\mathfrak{C}$.

DEFINITION. Let $A \subseteq \mathfrak{C}$, and let $N > 0$. Let $\mathcal{T}$ be a real-valued relation on $\mathscr{B}_N$. We say that $\mathcal{T}$ is *definable over* A if for every pair of rational numbers $r < s$ there exists a pair of positive bounded $L(A)$-formulas

$$\theta_1^{r,s}(\bar{x}) < \theta_2^{r,s}(\bar{x})$$

of the same arity as $\mathcal{T}$, such that for every $\bar{a} \in \mathscr{B}_N$,

$$\begin{array}{ccc} \mathcal{T}(\bar{a}) \leq r & \text{implies} & \models \theta_1^{r,s}(\bar{a}) \\ \models \theta_2^{r,s}(\bar{a}) & \text{implies} & \mathcal{T}(\bar{a}) \leq s. \end{array}$$

The map

$$(r, s) \mapsto (\, \theta_1^{r,s}, \, \theta_2^{r,s} \,)$$

is called a *definition schema* for $\mathcal{T}$ over A.

If $\mathcal{R}$ is a real-valued relation on $\mathfrak{C}$, we say that $\mathcal{R}$ is *definable over* A if $\mathcal{R} | \mathscr{B}_N$ is definable over A, for every $N > 0$.

Intuitively, $\mathcal{T}$ is definable over A if its values can be uniformly approximated at will by positive bounded $L(A)$-formulas.

By assumption, all real-valued relations are bounded. Thus, if $\mathcal{R}$ is an m-ary real-valued relation on $\mathfrak{C}$, then $\mathcal{R}$ is an element of the Banach space $\ell^\infty(\mathfrak{C}^m)$ and $\mathcal{R} | \mathscr{B}_N$ is an element of the Banach space $\ell^\infty(\, (\mathscr{B}_N)^m \,)$, for every $N > 0$.

PROPOSITION 1. *Let $A \subseteq \mathfrak{C}$, m be a positive integer, and $N > 0$. The set*

$$\mathrm{defn}(\mathscr{B}_N) = \{\, \mathcal{R} | \mathscr{B}_N \ : \mathcal{R} \textit{ is an } m\textit{-ary real-valued relation and } \mathcal{R} | \mathscr{B}_N \textit{ is definable over } A \,\}$$

is a closed subset of $\ell^\infty(\, (\mathscr{B}_N)^m \,)$.

PROOF: Let $\mathcal{T}$ be in the closure of $\mathrm{defn}(\mathscr{B}_N)$. Then $\mathcal{T}$ is a real-valued relation on $\mathscr{B}_N$. We prove that $\mathcal{T}$ is definable over A. Fix rational numbers $r < s$. We now describe how to find $L(A)$-formulas

$$\psi_1^{r,s}(\bar{x}) < \psi_2^{r,s}(\bar{x})$$

such that for $\bar{x} \in \mathscr{B}_N$,

$$(*) \qquad \begin{array}{ccc} \mathcal{T}(\bar{x}) \leq r & \text{implies} & \models \psi_1^{r,s}(\bar{x}) \\ \models \psi_2^{r,s}(\bar{x}) & \text{implies} & \mathcal{T}(\bar{x}) \leq s. \end{array}$$

First, take rational numbers r_1, r_2 such that $r < r_1 < r_2 < s$. Find now $\mathcal{R}|\mathscr{B}_N$ in $\mathrm{defn}(\mathscr{B}_N)$ such that

$$\| \, \mathcal{T} \; - \; \mathcal{R}|\mathscr{B}_N \, \|_\infty \leq \min\{\, r_1 - r, s - r_2 \,\}.$$

Let

$$(r,s) \mapsto (\, \theta_1^{r,s} \, , \, \theta_2^{r,s} \,)$$

be a definition schema for $\mathcal{R}|\mathscr{B}_N$. It is now easy to see that $(*)$ follows with $\psi_1^{r,s} \, : \, \theta_1^{r_1,r_2}$ and $\psi_2^{r,s} \, : \, \theta_2^{r_1,r_2}$. ∎

If $A \subseteq \mathfrak{C}$, an A-*automorphism* of $\mathfrak{C}$ is an automorphism of $\mathfrak{C}$ which fixes A pointwise.

DEFINITION. Let $A \subseteq \mathfrak{C}$ and $N > 0$. Let $\mathcal{T}$ be a real-valued relation on $\mathscr{B}_N$. An A-*conjugate* of $\mathcal{T}$ is a partial function of the form $f(\mathcal{T})$, where f is an A-automorphism of $\mathfrak{C}$. The set of A-conjugates of $\mathcal{T}$ is denoted $\mathrm{conj}^A(\mathcal{T})$.

Clearly, an A conjugate of a real-valued relation $\mathcal{T}$ on $\mathscr{B}_N$ is a real-valued relation on $\mathscr{B}_N$.

PROPOSITION 2. *Suppose that $\mathcal{R}|\mathscr{B}_N$ is an m-ary relation definable over A. Then $\mathrm{conj}^A(\mathcal{R}|\mathscr{B}_N)$ is a closed subset of $\ell^\infty(\,(\mathscr{B}_N)^m\,)$.*

PROOF: The proof is divided into steps. The end of the proof of each step is indicated by a square: □.

Let

$$(r,s) \mapsto (\; \theta_1^{r,s}(\bar{x}, \bar{a}_{r,s}), \;\; \theta_2^{r,s}(\bar{x}, \bar{a}_{r,s}) \;)$$

be a definition schema for $\mathcal{R}|\mathscr{B}_N$, where all the parameters of $\theta_1^{r,s}$ and $\theta_2^{r,s}$ are included in the tuple $\bar{a}_{r,s}$. If f is an A-automorphism of $\mathfrak{C}$, then

$$(r,s) \mapsto (\; \theta_1^{r,s}(\bar{x}, f(\bar{a}_{r,s})), \;\; \theta_2^{r,s}(\bar{x}, f(\bar{a}_{r,s})) \;)$$

is a definition schema for $f(\mathcal{R}|\mathscr{B}_N)$.

Take an element $\mathcal{T}$ of the closure of $\mathrm{conj}^A(\mathcal{R}|\mathscr{B}_N)$. Then $\mathcal{T}$ is a real-valued relation on $\mathscr{B}_N$. We wish to prove that $\mathcal{T}$ is an A-conjugate of $\mathcal{R}|\mathscr{B}_N$.

By Proposition 1, $\mathcal{T}$ is definable over A. Let

$$(r,s) \mapsto (\ \psi_1^{r,s}(\bar{x}),\ \psi_2^{r,s}(\bar{x})\ )$$

be a definition schema for $\mathcal{T}$.

For each pair of rational numbers $s < r$, pick r_1, r_2, r_3 and r_4 such that

$$r < r_1 < r_2 < r_3 < r_4 < s.$$

Define

$$\begin{aligned} \sigma^{r,s}(\bar{y}_{r_2,r_3}): \quad & \forall \bar{x}\ [\ \|\bar{x}\| \le N\ \rightarrow\ (\ neg(\psi_2^{r,r_1}(\bar{x}))\ \vee\ \theta_1^{r_2,r_3}(\bar{x},\bar{y}_{r_2,r_3})\ )\ ] \\ & \wedge \forall \bar{x}\ [\ \|\bar{x}\| \le N\ \rightarrow\ (\ neg(\theta_2^{r_2,r_3}(\bar{x},\bar{y}_{r_2,r_3}))\ \vee\ \psi_1^{r_4,s}(\bar{x})\ )\ ]. \end{aligned}$$

STEP 1. *Let $s < r$, and let f be an A-automorphism of $\mathfrak{C}$. If*

$$\|\ f(\mathcal{R}|\mathscr{B}_N)\ -\ \mathcal{T}\ \|_\infty \le \min\{\ r_2 - r_1, r_4 - r_3\ \},$$

then $\models \sigma^{r,s}(f(\bar{a}_{r_2,r_3}))$.

PROOF OF STEP 1: Fix $\bar{x} \in \mathscr{B}_N$. We have

$$\models \psi_2^{r,r_1}(\bar{x}) \qquad \text{implies} \qquad \mathcal{T}(\bar{x}) \le r_1.$$

Since $\|\ f(\mathcal{R}|\mathscr{B}_N)\ -\ \mathcal{T}\ \|_\infty \le r_2 - r_1$,

$$\mathcal{T}(\bar{x}) \le r_1 \qquad \text{implies} \qquad \models \theta^{r_2,r_3}(\bar{x}, f(\bar{a}_{r_2,r_3})).$$

From these two implications we conclude

$$(*) \qquad \models\ neg(\psi_2^{r,r_1}(\bar{x}))\ \vee\ \theta^{r_2,r_3}(\bar{x}, f(\bar{a}_{r_2,r_3})).$$

Since $\|\ f(\mathcal{R}|\mathscr{B}_N)\ -\ \mathcal{T}\ \|_\infty \le r_4 - r_3$,

$$\theta_2^{r_2,r_3}(\bar{x}, f(\bar{a}_{r_2,r_3})) \qquad \text{implies} \qquad \mathcal{T}(\bar{x}) \le r_4.$$

Also,

$$\mathcal{T}(\bar{x}) \le r_4 \qquad \text{implies} \qquad \psi_2^{r_4,s}(\bar{x}).$$

From the preceding two implications we conclude

$$(**) \qquad \models\ neg(\theta_2^{r_2,r_3}(f(\bar{x}, \bar{a}_{r_2,r_3})))\ \vee\ \psi_2^{r_4,s}(\bar{x}).$$

Since $\bar{x}$ is arbitrary in $\mathscr{B}_N$, the claim of Step 1 follows from $(*)$ and $(**)$.$\square$

Define

$$\begin{aligned} \Sigma &= \{\ \ \sigma^{r,s}(\bar{y}_{r_2,r_3})\ \ :\ \ r < s\ \ \} \\ \Gamma &= \{\ \ \gamma(\bar{y}_{r_2,r_3})\ :\ \gamma \in tp(\bar{a}_{r_2,r_3}|A)\ \ \}. \end{aligned}$$

STEP 2. $\Sigma \cup \Gamma$ *is consistent.*

PROOF OF STEP 2: Take

$$\sigma^{r^1,s^1}(\bar y_{r_2^1,r_3^1}),\ldots,\sigma^{r^k,s^k}(\bar y_{r_2^k,r_3^k})$$

in Σ, and

$$\gamma_1(\bar y_{r_2^k,r_3^k}),\ldots,\gamma_k(\bar y_{r_2^k,r_3^k})$$

in Γ, in order to prove that the conjunction

$$\bigwedge_{i=1}^{k}\sigma^{r^i,s^i}(\bar y_{r_2^i,r_3^i})\wedge\bigwedge_{i=1}^{k}\gamma_i(\bar y_{r_2^i,r_3^i})$$

is consistent.

Take an A-conjugate $f(\mathcal{R}|\mathscr{B}_N)$ of $\mathcal{R}|\mathscr{B}_N$ such that

$$\|\ f(\mathcal{R}|\mathscr{B}_N)\ -\ \mathcal{T}\ \|_\infty\leq\min_i\{\,r_2^i-r_1^i,r_4^i-r_3^i\,\}$$

Then, by Step 1,

$$\models\ \bigwedge_{i=1}^{k}\sigma^{r^i,s^i}(\bar a_{r_2^i,r_3^i})\wedge\bigwedge_{i=1}^{k}\gamma_i(\bar a_{r_2^i,r_3^i}).\Box$$

Let now $(\,c_{r_2,s_3}\ :\ r<s\,)$ realize $\Sigma\cup\Gamma$. For each pair $r<s$ pick a formula $\theta^{r,s}_{\frac{3}{2}}(\bar y,\bar x)$ such that

$$\theta_1^{r,s}<\theta^{r,s}_{\frac{3}{2}}<\theta_2^{r,s}.$$

STEP 3. *The map*

$$(r,s)\mapsto(\ \theta_1^{r,s}(\bar x,\bar c_{r_2,r_3}),\ \theta^{r,s}_{\frac{3}{2}}(\bar x,\bar c_{r_2,r_3})\ )$$

is a definition schema for $\mathcal{T}$.

PROOF OF STEP 3: Fix $\bar x\in\mathscr{B}_N$. We have

$$\mathcal{T}(\bar x)\leq r\qquad\text{implies}\qquad\not\models neg(\psi_2^{r,r_1}(\bar x)).$$

Hence, since the $\bar c_{r,s}$'s realize Σ,

(*) $$\mathcal{T}(\bar x)\leq r\qquad\text{implies}\qquad\models\theta_1^{r_2,r_3}(\bar x,\bar c_{r_2,r_3}).$$

Also,

$$\models\theta^{r_2,r_3}_{\frac{3}{2}}(\bar x,\bar c_{r_2,r_3})\qquad\text{implies}\qquad\not\models neg(\theta_2^{r_2,r_3}(\bar x,\bar c_{r_2,r_3})).$$

Since the $\bar c_{r,s}$'s realize Σ,

(**) $$\models\theta^{r_2,r_3}_{\frac{3}{2}}(\bar x,\bar c_{r_2,r_3}))\qquad\text{implies}\qquad\mathcal{T}(\bar x)\leq s.$$

Step 3 follows from $(*)$ and $(**)$.$\Box$

STEP 4. *The map*

$$(r,s) \mapsto (\ \theta_1^{r_2,r_3}(\bar{a}_{r_2,r_3},\bar{x}),\ \theta_{\frac{3}{2}}^{r_2,r_3}(\bar{a}_{r_2,r_3},\bar{x})\ )$$

is a definition schema for $\mathcal{R}|\mathscr{B}_N$.

PROOF OF STEP 4: Clear.$\square$

Now we finish the proof of the proposition: Since the $\bar{c}_{r,s}$'s realize Γ, there is an A-automorphism g of $\mathfrak{C}$ such that $g(\bar{a}_{r,s}) = \bar{c}_{r,s}$, for $r < s$. By the last two steps, $g(\mathcal{R}|\mathscr{B}_N) = \mathcal{T}$. Thus $\mathcal{T}$ is an A-conjugate of $\mathcal{R}|\mathscr{B}_N$. $\blacksquare$

3. Definability and conjugates: the easy case

Here we characterize definability of $\mathcal{R}$ in terms of the set *conjugates* of $\mathcal{R}$.

DEFINITION. If $\mathcal{R}$ is a real-valued relation, an *A-conjugate* of $\mathcal{R}$ is a real valued-relation of the form $f(\mathcal{R})$, where f is an A-automorphism of $\mathfrak{C}$.

THEOREM 3. *A real-valued relation $\mathcal{R}$ on $\mathfrak{C}$ is definable over A if and only $\mathcal{R}$ is its only A-conjugate.*

PROOF: This follows directly from the Beth definability theorem for positive bounded formulas. See [10, Section 1.4]. $\blacksquare$

The original question was to characterize definability of $\mathcal{R}$ in nonlogical terms. The above characterization is in principle logical, since it involves the model $\mathfrak{C}$, whose existence is proved by model theoretical means. However, if κ is an infinite cardinal, κ-saturated normed space models of a positive bounded theory can be realized as functional analytic ultrapowers, using κ^+-good ultrafilters. This is proved as is classical model theory. (See [15].) This allows us to restate Theorem 3 as follows.

COROLLARY 4. *Let E be a model of T. Let $\mathcal{R}$ be a real-valued relation on E and let A be a subset of E. Then the following conditions are equivalent.*

(1) *$\mathcal{R}$ is definable over A;*
(2) *There exists an ultrapower $(\mathfrak{C}, \mathfrak{R})$ of $(E, \mathcal{R})$ such that $\mathfrak{R}$ is its only conjugate in $\mathfrak{C}$.*

4. Definability and conjugates: the general case

Theorem 3 motivates the following definition.

DEFINITION. Let $A \subseteq \mathfrak{C}$ and $N > 0$. Let $\mathcal{T}$ be an m-ary real-valued relation on $\mathscr{B}_N$. We say that $\mathcal{T}$ is *almost over A* if $\mathrm{conj}^A(\mathcal{T})$ is a compact subset of $\ell^\infty(\,(\mathscr{B}_N)^m\,)$.

A real-valued relation $\mathcal{R}$ on $\mathfrak{C}$ is *almost over A* if $\mathcal{R}|\mathscr{B}_N$ is almost over A, for every $N > 0$.

When the metrics are discrete, metric space structures correspond to the kinds of structures studied in first-order model theory, real-valued relation correspond to relations in the ordinary sense, and the model theory of metric space structures collapses to first-order model theory. Under these restrictions, the preceding notion coincides with the classical notion of "almost over A" introduced by S. Shelah. See [15].

THEOREM 5. *Let $A \subseteq \mathfrak{C}$. Let $\mathcal{R}$ be an m-ary real-valued relation on $\mathfrak{C}$. The following conditions are equivalent.*

(1) *$\mathcal{R}$ is almost over A.*
(2) *For every $N > 0$ there exists a pseudometric ρ on $(\mathscr{B}_N)^m$ such that*
  (i) *$\mathcal{R}|\mathscr{B}_N$ is uniformly continuous with respect to ρ;*
  (ii) *$(\mathscr{B}_N)^m$ is compact with respect to ρ;*
  (iii) *ρ is definable over A.*
(3) *$\mathcal{R}$ is definable over every model containing A.*
(4) *$\mathcal{R}$ is definable, and $\operatorname{card}(\operatorname{conj}^A(\mathcal{R})) \leq 2^{\aleph_0}$.*
(5) *$\mathcal{R}$ is definable, and there exists a cardinal κ such that $\operatorname{card}(\operatorname{conj}^A(\mathcal{R})) \leq \kappa$.*
(6) *$\mathcal{R}$ is definable, and $\operatorname{conj}^A(\mathcal{R}|\mathscr{B}_N)$ is a precompact subset of $\ell^\infty((\mathscr{B}_N)^m)$, for every $N > 0$.*

PROOF: (1) $\Rightarrow$ (2): For $\bar{x}, \bar{y} \in (\mathscr{B}_N)^m$ define

$$\rho(\bar{x}, \bar{y}) = \sup\{\, |\mathcal{T}(\bar{x}) - \mathcal{T}(\bar{y})| \;:\; \mathcal{T} \in \operatorname{conj}^A(\mathcal{R}|\mathscr{B}_N)\,\}.$$

It is easy to see that ρ is a pseudometric. Also,

$$\mathcal{R}|\mathscr{B}_N(\bar{x}, \bar{y}) \leq \rho(\bar{x}, \bar{y}).$$

Hence $\mathcal{R}|\mathscr{B}_N$ is uniformly continuous with respect to ρ.

We prove that $\mathscr{B}_N$ is compact with respect to ρ. Suppose that this is not the case. Then there exist $\epsilon > 0$ and a sequence $(\bar{a}_k)$ in $(\mathscr{B}_N)^m$ such that $\rho(\bar{a}_k, \bar{a}_l) > \epsilon$ for $k < l$. By the definition of ρ, for each pair $k < l$ there exists an A-conjugate $\mathcal{T}_{k,l}$ of $\mathcal{R}|\mathscr{B}_N$ such that

$$|\,\mathcal{T}_{k,l}(\bar{a}_k) - \mathcal{T}_{k,l}(\bar{a}_l)\,| > \epsilon, \qquad \text{for } k < l.$$

Now, since $\mathcal{R}|\mathscr{B}_N$ is almost over A, there exist finitely many A-conjugates of $\mathcal{R}|\mathscr{B}_N$, say, $\mathcal{S}_1, \ldots, \mathcal{S}_t$, such that each $\mathcal{T}_{k,l}|\mathscr{B}_N$ is within $\frac{\epsilon}{3}$ of some $\mathcal{S}_i$ in $\ell^\infty((\mathscr{B}_N)^m)$. By replacing $(\bar{a}_k)$ by a subsequence, we may assume that each $\mathcal{T}_{k,l}|\mathscr{B}_N$ is within $\frac{\epsilon}{3}$ of *the same* $\mathcal{S}_i$. Thus, we find an A-conjugate $\mathcal{S}$ of $\mathcal{R}|\mathscr{B}_N$ such that

$$|\,\mathcal{T}_{k,l}(\bar{a}_k) - \mathcal{S}(\bar{a}_k)\,| < \frac{\epsilon}{3}, \qquad \text{for all } k.$$

We have then

$$|\,\mathcal{S}(\bar{a}_k) - \mathcal{S}(a_l)\,| > \frac{\epsilon}{3}, \qquad \text{for } k < l.$$

But this is a contradiction, for $\mathcal{R}$ (and hence $\mathcal{S}$) takes values on the bounded interval $[0,1]$.

It remains to prove that ρ is definable over A. By Theorem 3, we only need to prove that $f(\rho) = \rho$ for every A-automorphism f of $\mathfrak{C}$. But if f is such an automorphism,

$$\{\ f(\mathcal{T})\ :\ \mathcal{T}\ \mathrm{conj}^A(\mathcal{R}|\mathscr{B}_N)\ \} = \mathrm{conj}^A(\mathcal{R}|\mathscr{B}_N).$$

Therefore, $f(\rho) = \rho$.

(2) $\Rightarrow$ (3): Let E be a model containing A. Fix $N > 0$, in order to prove that $\mathcal{R}|\mathscr{B}_N$ is definable over E.

Fix $r < s$ in $[0,1]$. We wish to find positive bounded $L(E)$-formulas $\theta_1^{r,s}(\bar{x}) < \theta_2^{r,s}(\bar{x})$ such that for every $\bar{x} \in \mathscr{B}_N$,

$$\begin{array}{ccc} \mathcal{R}(\bar{x}) \le r & \text{implies} & \models \theta_1^{r,s}(\bar{x}) \\ \models \theta_2^{r,s}(\bar{x}) & \text{implies} & \mathcal{R}(\bar{x}) \le s. \end{array}$$

Take $N'' > N' > N$. Let ρ be a pseudometric as in (2), but with N'' in place of N. Let

$$(r,s) \mapsto (\ \psi_1^{r,s},\ \psi_2^{r,s}\ )$$

be a definition schema for ρ definable over A.

Fix $\delta > 0$ such that on $\mathscr{B}_{N''}$,

$$\|\bar{x} - \bar{y}\| \le \delta \qquad \text{implies} \qquad |\,\mathcal{R}(\bar{x}) - \mathcal{R}(\bar{y})\,| \le s - r.$$

Since $\mathscr{B}_{N'}$ is precompact with respect to ρ, there exist $\bar{c}_1, \dots, \bar{c}_s \in \mathscr{B}_{N'}$ such that every element of $(\mathscr{B}_{N'})^m$ is within $\frac{\delta}{8}$ of some $\bar{c}_i$ (with respect to ρ). Thus,

$$\models \forall \bar{x}\ (\ \|\bar{x}\| \le N'\ \rightarrow\ \bigvee_{i=1}^{s} \psi_1^{\frac{\delta}{8},\frac{\delta}{4}}(\bar{x}, \bar{c}_i)\ ).$$

Now, since $E \prec_{\mathcal{A}} \mathfrak{C}$, there exist $\bar{b}_1, \dots, \bar{b}_s \in \mathscr{B}_{N''}(E)$ such that

$$\models \forall \bar{x}\ (\ \|\bar{x}\| \le N\ \rightarrow\ \bigvee_{i=1}^{s} \psi_2^{\frac{\delta}{8},\frac{\delta}{4}}(\bar{x}, \bar{b}_i)\ ).$$

Hence, every element of $(\mathscr{B}_N)^m$ is within $\frac{\delta}{4}$ of some $\bar{b}_i$. Find $\bar{a}_1, \ldots, \bar{a}_t$ from among the $\bar{b}_i$'s such that every element of $(\,\mathcal{R}^{-1}([0,r])\,) \cap \mathscr{B}_N$ is within $\frac{\delta}{4}$ of some $\bar{a}_i$. Then, for every $\bar{x} \in (\mathscr{B}_N)^m$,

$$\mathcal{R}(\bar{x}) \leq r \qquad \text{implies} \qquad \models \bigvee_{i=1}^{t} \psi_1^{\frac{\delta}{4},\frac{\delta}{2}}(\bar{x}, \bar{a}_i)$$

$$\models \bigvee_{i=1}^{t} \psi_2^{\frac{\delta}{4},\frac{\delta}{2}}(\bar{x}, \bar{a}_i). \qquad \text{implies} \qquad \mathcal{R}(\bar{x}) \leq s.$$

Thus, $(*)$ follows by taking

$$\theta_1^{r,s}(\bar{x}) \; : \; \bigvee_{i=1}^{t} \psi_1^{\frac{\delta}{4},\frac{\delta}{2}}(\bar{x}, \bar{a}_i)$$

$$\theta_2^{r,s}(\bar{x}) \; : \; \bigvee_{i=1}^{t} \psi_2^{\frac{\delta}{4},\frac{\delta}{2}}(\bar{x}, \bar{a}_i).$$

(3) $\Rightarrow$ (4): Fix a separable model E such that $\mathcal{R}|\mathscr{B}_N$ is definable over E. We claim that every A-conjugate of $\mathcal{R}|\mathscr{B}_N$ is also definable over E. This will prove (4), for then every A-conjugate of $\mathcal{R}|\mathscr{B}_N$ is determined by countably many formulas over E, and there are at most $2^{\aleph_0}$ such formulas.

Let $\mathcal{T}$ be an A-conjugate of $\mathcal{R}|\mathscr{B}_N$. Find an A-automorphism f of $\mathfrak{C}$ such that $\mathcal{T} = f(\mathcal{R}|\mathscr{B}_N)$. By assumption, $\mathcal{R}|\mathscr{B}_N$ is definable over the model $f^{-1}(E)$. Thus, $\mathcal{T}$ is definable over $f(f^{-1}(E)) = E$.

(4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (6): Let

$$(r,s) \mapsto (\; \theta_1^{r,s}(\bar{x}, f(\bar{a}_{r,s})),\; \theta_2^{r,s}(f(\bar{x}, \bar{a}_{r,s}))\;)$$

be a definition schema for $\mathcal{R}|\mathscr{B}_N$.

Suppose that the set of A-conjugates of $\mathcal{R}|\mathscr{B}_N$ is not precompact. Then there exist $\epsilon > 0$ and A-conjugates $(\mathcal{T}_k)_{k<\omega}$ of $\mathcal{R}|\mathscr{B}_N$ such that

$$\|\, \mathcal{T}_k \;-\; \mathcal{T}_l \,\|_\infty > \epsilon, \qquad \text{for } k < l.$$

For each pair $k < l$ there exists $\bar{b}_{k,l} \in \mathscr{B}_N$ such that

$$(*) \qquad |\, \mathcal{T}_k(\bar{b}_{k,l}) - \mathcal{T}_l(\bar{b}_{k,l}) \,| > \epsilon.$$

By replacing $(\mathcal{T}_k)_{k<\omega}$ by a subsequence if necessary, we may assume that the sequences $(\mathcal{T}_k(\bar{b}_{k,l}))_{k<\omega}$ and $(\mathcal{T}_l(\bar{b}_{k,l}))_{l<\omega}$ converge in $[0,1]$, say,

$$\lim_{k\to\infty} \mathcal{T}_k(\bar{b}_{k,l}) = \alpha$$

$$\lim_{l\to\infty} \mathcal{T}_l(\bar{b}_{k,l}) = \beta.$$

By $(*)$, $\alpha \neq \beta$. Assume $\alpha < \beta$ (the case $\alpha > \beta$ is analogous). For each pair of rational numbers $r < s$, take rational numbers r, r', r'', r''' and s such that

$$\alpha < r < r' < r'' < r''' < s < \beta.$$

Then

$$(**) \qquad \begin{aligned} \mathcal{T}_k(\bar{b}_{k,l}) &\leq r, && \text{for large } k \\ \mathcal{T}_l(\bar{b}_{k,l}) &\geq s, && \text{for large } l. \end{aligned}$$

Let κ be a cardinal as in (5). Define

$$\begin{aligned} \Sigma(x_{i,j}, \bar{y}_{r,s,i}) &= \{ \ \theta_1^{r,r'}(x_{i,j}, \bar{y}_{r,s,j}) \wedge neg(\theta_2^{r'',r'''}(x_{r,s}, \bar{y}_{r,s,j})) \ : \ \ r < s, \ \ i < j < \kappa^+ \ \} \\ \Gamma(\bar{y}_{r,s,i}) &= \{ \ \ \gamma(\bar{y}_{r,s,i}) \ \ : \ \ \gamma \in tp(\bar{a}_{r,s}|A), \ \ r < s, \ \ i < \kappa^+, \}. \end{aligned}$$

By $(**)$, $\Sigma \cup \Gamma$ is consistent. By the saturation of $\mathfrak{C}$, we can find $(\ \bar{a}'_{r,s,i} \ : \ r < s, \ i < \kappa^+ \ )$ and $(\ \bar{b}'_{i,j} \ : \ i < j < \kappa^+ \ )$ such that

$$\models_{\mathcal{A}} \Sigma(b'_{i,j}, \bar{a}'_{r,s,i}) \cup \Gamma(\bar{a}'_{r,s,i}), \qquad \text{for } r < s \text{ and } i < j < \kappa.$$

By the homogeneity of $\mathfrak{C}$, there exists an A-automorphism f_i of $\mathfrak{C}$ mapping $\bar{a}_{r,s}$ to $\bar{a}'_{r,s,i}$. Thus, $f_i(\mathcal{R}|\mathscr{B}_N)$ is an A-conjugate of $\mathcal{R}|\mathscr{B}_N$ and

$$(r,s) \mapsto (\ \theta_1^{r,s}(\bar{x}, \bar{a}'_{r,s,i}), \ \theta_2^{r,s}(\bar{x}, \bar{a}'_{r,s,i}) \ )$$

is a definition schema for $f_{r,s,i}(\mathcal{R}|\mathscr{B}_N)$. But then, for $r < s$ and $i < j < \kappa^+$,

$$\begin{aligned} f_i(\mathcal{R})(\bar{b}'_{i,j}) &\leq r' \\ f_j(\mathcal{R})(\bar{b}'_{i,j}) &\geq r''. \end{aligned}$$

Thus, $f_i \neq f_j$ for $i < j < \kappa$. This is in contradiction with (5).

(6) $\Rightarrow$ (1) is given by Corollary 2. ■

COROLLARY 6. *Let E be a model of T. Let $\mathcal{R}$ be an m-ary real-valued relation on E and let A be a subset of E. Then the following conditions are equivalent.*

(1) *If $(E', \mathcal{R}', a)_{a\in A}$ is elementary equivalent to $(E, \mathcal{R}, a)_{a\in A}$, then $\mathcal{R}'$ is definable over E';*

(2) *There exists an ultrapower $(\mathfrak{C}, \mathfrak{R})$ of $(E, \mathcal{R})$ such that $\mathfrak{R}$ is definable over every model F satisfying $A \subseteq F \subseteq \mathfrak{C}$;*

(3) *There exists an ultrapower $(\mathfrak{C}, \mathfrak{R})$ of $(E, \mathcal{R})$ satisfying the following condition: for every $N > 0$ there exists a pseudometric ρ on $(\mathscr{B}_N(\mathfrak{C}))^m$ such that*

  (i) *$\mathfrak{R}|\mathscr{B}_N(\mathfrak{C})$ is uniformly continuous with respect to ρ;*

  (ii) *$(\mathscr{B}_N(E))^m$ is compact with respect to ρ;*

  (iii) *ρ is its only A-conjugate in $\mathfrak{C}$.*

5. Definable types

Let $A \subseteq B \subseteq \mathfrak{C}$. A type $p(\bar{x}) \in S_n(B)$ is called *definable over A* if for every pair of L-formulas

$$\phi(\bar{x}, \bar{y}) < \psi(\bar{x}, \bar{y})$$

there exists a pair of $L(A)$-formulas

$$\theta_1^{\phi,\psi}(\bar{y}) < \theta_2^{\phi,\psi}(\bar{y})$$

such that for all $\bar{a} \in \mathscr{B}(\mathbb{Q}B)$,

$$\begin{aligned} \phi(\bar{x}, \bar{a}) \in p(\bar{x}) \qquad &\text{implies} \qquad \models_{\mathcal{A}} \theta_1^{\phi,\psi}(\bar{a}) \\ \models_{\mathcal{A}} \theta_2^{\phi,\psi}(\bar{a}) \qquad &\text{implies} \qquad \psi(\bar{x}, \bar{a}) \in p. \end{aligned}$$

The map

$$(\phi, \psi) \mapsto (\, \theta_1^{\phi,\psi} \,,\, \theta_2^{\phi,\psi} \,)$$

is called a *definition schema* for p over A.

In [10] we proved that a theory T is stable if every type consistent with T is definable.

Let E be a model and let $p(\bar{x}) \in S_n(E)$. For each L-formula $\sigma(\bar{x}, \bar{y})$ with $\ell(\bar{y}) = m$ we define a m-ary real-valued relation

$$\mathcal{P}_\sigma : E^m \to [0, \infty]$$

such that

$$\sigma(\bar{x}, \bar{a}) \in p \qquad \text{if and only if} \qquad \mathcal{P}_\sigma(\bar{a}) = 0.$$

The construction is as follows. For each formula $\sigma(\bar{x}, \bar{y})$ we take a set of approximations of σ

$$\{\, \sigma_r(\bar{x}, \bar{y}) \;:\; r \in \mathbb{Q}^+ \,\}$$

such that

- $\sigma_r < \sigma_s$ if and only if $r < s$;
- $\sigma_r \to \sigma$ as $r \to 0$, i.e., for every $\tau > \sigma$ there exists $q \in \mathbb{Q}$ such that $\sigma < \sigma_r$ for $0 < r < q$.

Define

$$\mathcal{P}_\sigma(\bar{a}) = \inf\{\, r \in \mathbb{Q}^+ \;:\; \sigma_r(\bar{x}, \bar{a}) \in p \,\}.$$

The real-valued relation $\mathcal{P}_\sigma$ is uniformly continuous on every bounded subset of E. We denote by $L(\mathcal{P}_\sigma : \sigma \in L)$ the expansion of the language L with predicates for the $\mathcal{P}_\sigma$'s and the moduli of uniform continuity provided for them by the Perturbation Lemma. The $L(\mathcal{P}_\sigma : \sigma \in L)$-structure $(E, \mathcal{P}_\sigma \,:\, \sigma \in L)$ is called a *p-morleyization* of E.

A type $p(\bar{x}) \in S_n(E)$ is uniquely determined by any p-morleyization $(E, \mathcal{P}_\phi)_{\phi(\bar{x}) \in L}$ of $\mathfrak{C}$. Clearly, p is definable over A if and only if, for each $\phi(\bar{x}) \in L$, the real-valued relation $\mathcal{P}_\phi$ is definable over A. This allows us to use the results from Sections 2 and 3 to characterize definability of types.

DEFINITION. Let Σ be a set of formulas over $\mathfrak{C}$. An *A-conjugate* of Σ is a set of the form $f(\Sigma)$, where f is an A-automorphism of $\mathfrak{C}$. The set of A-conjugates of Σ is denoted $\operatorname{conj}^A(\Sigma)$.

COROLLARY 7. *A type p over $\mathfrak{C}$ is definable over A if and only if p is its only A-conjugate.*

PROOF: Immediate from Theorem 3 and the preceding remarks. ■

DEFINITION. Let $p(\bar{x})$ and $q(\bar{x})$ be types over $\mathfrak{C}$. Let

$$(\mathfrak{C}, \mathcal{P}_\phi)_{\phi(\bar{x})\in L} \qquad\qquad (\mathfrak{C}, \mathcal{Q}_\phi)_{\phi(\bar{x})\in L}$$

corresponding morleyizations. For $N > 0$, we define

$$\operatorname{dist}(\ p|\mathscr{B}_N,\ q|\mathscr{B}_N\ ) = \sup\{\ \|\,\mathcal{P}_\phi|\mathscr{B}_N\ -\ \mathcal{Q}_\phi|\mathscr{B}_N\,\|_\infty\ :\ \phi(\bar{x}) \in L\ \}.$$

Then dist is a metric on $S(\mathscr{B}_N)$.

DEFINITION. A type p over $\mathfrak{C}$ is *almost over A* if $\operatorname{conj}^A(p)$ is a compact subset of $(\ S(\mathscr{B}_N),\ \operatorname{dist}\ )$, for every $N > 0$.

In principle, being almost over A depends of the particular choice of morleyizations in the definition of dist. However, it follows from the following corollary that this is not the case.

COROLLARY 8. *Let $p \in S(\mathfrak{C})$. The following conditions are equivalent.*

(1) *p is almost over A.*
(2) *p is definable over every model containing A.*
(3) *p is definable, and* $\operatorname{card}(\operatorname{conj}^A(p)) \le 2^{\aleph_0}$.
(4) *p is definable, and there exists a cardinal κ such that* $\operatorname{card}(\operatorname{conj}^A(p)) \le \kappa$.

PROOF: Immediate from Theorem 5. ■

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