Ultraproducts in Analysis

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Introduction

The ideas and methods of model theory are being applied today in nearly all parts of mathematics. Here we concentrate on a framework for applications in functional analysis. Model theory has already provided several tools for the research analyst, of which the most important are: (a) the Banach space *ultraproduct* and *nonstandard hull* constructions in functional analysis; (b) spaces of (model-theoretic) *types* as used in the geometry of Banach spaces; and (c) *Loeb measure spaces* in stochastic analysis and its applications. In this paper we explain a systematic model theoretic framework within which these tools (especially (a) and (b)) are naturally situated. Our main intended audience consists of analysts who are familiar with the ultraproduct construction, and this perspective has strongly influenced our presentation of the material. We expect that many model theorists will also find something of interest in this subject. In particular, we indicate the initial steps of a program for introducing the key ideas and methods of model theory into functional analysis in a systematic and comprehensive way.

Applications of model theory in functional analysis have been pursued since the mid 1960s, beginning with the introduction of Banach space ultraproducts by Bretagnolle, Dacunha-Castelle, and Krivine [BDCK66] [Kri67] [DCK72] and nonstandard hulls by Luxemburg [Lux69b]. These two constructions were first used at about the same time and they are essentially the same; however, these initial steps led to largely independent lines of research that have still not been fully integrated.

In nonstandard analysis, two important publications in the late 1960s were Abraham Robinson's book [Rob66] and the conference proceedings [Lux69a]. These books showed a wide range of areas within analysis in

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which one could expect fruitful applications of model theory, and they are still of value.

Robinson's initial goal was to provide a rigorous foundation for using infinitesimals in analysis. Banach space ultraproducts were introduced to investigate $L_p(\mu)$-spaces and related spaces. These methods have evolved considerably and they are now being used across the entire spectrum of mathematical analysis.

An up to date presentation of many applications to which Robinson's ideas have led, many of which are in functional analysis, can be found in [ACH97]. That book is intended to be accessible to all mathematicians, and includes introductory material on nonstandard analysis. Surveys of how ultraproducts and nonstandard analysis have been applied in functional analysis include [Hei80], [Sim82], and [HM83b].

There is a definite thought process that leads from (Banach space) ultraproducts to the model theory that is presented here, as we now indicate. Noting that ultraproducts are widely and effectively used in functional analysis, it becomes natural to analyze, as precisely as possible, what a normed space structure has in common with its ultrapowers. This problem can be formulated in an abstract way as is done in the next paragraphs. (For convenience we use some of the terminology that is introduced later in the paper.)

Given two normed space structures $\mathcal{M}$ and $\mathcal{N}$ with the same signature, let us consider them to be *ultrapower equivalent* if there exist ultrafilters D and E for which the ultrapowers $(\mathcal{M})_D$ and $(\mathcal{N})_E$ are isomorphic. (Since norms are included as distinguished functions in the structures we consider, every isomorphism of normed space structures is necessarily isometric.) It is easy to see that $\mathcal{M}$ is ultrapower equivalent to any of its ultrapowers.

It is clear that the relation of ultrapower equivalence is reflexive and symmetric; eventually we prove that it is transitive, but this is far from clear at the beginning. Therefore we introduce the transitive closure: define $\mathcal{M} \sim \mathcal{N}$ to mean that there is a finite sequence of normed space structures $\mathcal{M}_1, \ldots, \mathcal{M}_n$ such that $\mathcal{M}_1 = \mathcal{M}$, $\mathcal{M}_n = \mathcal{N}$, and for each $k = 1, \ldots, n-1$ we have that $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$ are ultrapower equivalent. In other words, $\sim$ is the coarsest equivalence relation on normed space structures under which (a) $\mathcal{M} \sim (\mathcal{M})_D$ for every structure $\mathcal{M}$ and every ultrafilter D, and (b) $\mathcal{M} \sim \mathcal{N}$ whenever $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Understanding what it is that normed space structures have in common with their ultrapowers is the same as understanding the equivalence relation $\sim$. Our analysis shows that $\sim$ is identical to a relation of *elementary equivalence* with respect to a certain logic for normed space structures. The formal sentences of this logic are the *positive bounded sentences* of a certain first order language. Elementary equivalence is taken with respect to a semantics of *approximate satisfaction*. (All of this is explained in Chapter 5.) Given two normed space structures $\mathcal{M}$ and $\mathcal{N}$ with the same signature, we write $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$ if, for every positive bounded sentence σ in their common signature, σ is approximately satisfied in $\mathcal{M}$ if and only if σ is approximately satisfied in $\mathcal{N}$. When this condition holds we say that $\mathcal{M}$ and $\mathcal{N}$ are *approximately elementarily equivalent*.

Our solution to the problem of understanding the equivalence relation $\sim$ is to show that it is identical to $\equiv_{\mathcal{A}}$. Indeed, it turns out that these relations are the same as the relation of ultrapower equivalence itself. The easy direction is Corollary 9.4: if $\mathcal{M}$ and $\mathcal{N}$ have isomorphic ultrapowers, then $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$. The converse, which has a difficult proof using combinatorial ideas going back to a related argument of Shelah [She71], is Theorem 10.7.

The model theory for normed space structures that is described here has turned out to have many unexpected features and to be a close analogue of ordinary first order model theory. A large part of the basic aspects of this theory is outlined here. This model theory has been developed since the mid 1970s (see especially [Hen75] and [Hen76]) and has been used from time to time in applications of nonstandard analysis (especially of the nonstandard hull construction) in functional analysis. By now it has shown itself to be the "correct" analogue of first order logic for the normed structures that arise in functional analysis. (In addition to the results presented here, see the Lindström-type maximality theorem for this logic proved by the second author [Iov01].)

Perhaps the most novel aspect of this theory is the relation $\models_{\mathcal{A}}$ of approximate satisfaction. (See Definition 5.9.) It is natural to ask how necessary or intrinsic to this setting the relation $\models_{\mathcal{A}}$ may be. One indication that $\models_{\mathcal{A}}$ is an essential ingredient of any model theory that is adequate for handling ultrapowers of normed space structures is given by Proposition 9.26. Suppose $\mathcal{M}$ is a normed space structure, σ is a positive bounded sentence in the signature of $\mathcal{M}$, and D is any countably incomplete ultrafilter. Then σ is approximately satisfied in $\mathcal{M}$ if and only if σ is satisfied (in the usual semantics of ordinary model theory) in the ultrapower $(\mathcal{M})_D$. (Corollary 9.3 shows that this is also equivalent to the statement that σ is *approximately* satisfied in $(\mathcal{M})_D$.) Note that this yields an alternative "soft" way to define approximate satisfaction for positive bounded sentences, in which approximations do not appear at all. The same equivalences are true for positive bounded formulas applied to elements of $\mathcal{M}$.

In order to cover as many aspects of this model theory as possible in a limited space, we have omitted many proofs. Usually the proofs being left out are reasonably routine, at least for mathematicians who have had some exposure to the basic parts of formal logic (first order logic). Often these proofs have already appeared somewhere in the literature, if usually in a somewhat more restricted framework than the one found here, and we give pointers to that literature.

We conclude this introduction with a summary of each chapter.

2. Normed Space Structures

The general class of structures to which our model theory applies is introduced. These are many-sorted structures, in which each sort is a normed space. The rest of the structure consists of a family of functions between various sorts; these functions must be uniformly continuous on each bounded subset of their domains. Most of the standard normed structures of functional analysis are of this type, including normed spaces, normed vector lattices, normed algebras, and the like.

3. Signatures

In order to treat normed space structures in a general setting, it is convenient to introduce *signatures* for them; these provide a basis for indexing the sorts and functions that make up each structure. These are also helpful in giving exact definitions of such concepts as *embedding* and *isomorphism* for normed space structures.

4. Ultrapowers of Normed Space Structures

The Banach space ultrapower construction is well known and widely used for normed spaces, and it extends in a completely natural way to the normed space structures that are considered here. Indeed, the requirement that all functions in a normed space structure are uniformly continuous (and hence bounded) on bounded sets is exactly what is needed to permit the ultrapower construction to be applied to such a structure.

5. Positive Bounded Formulas

Here we introduce the formal language of *positive bounded formulas* in the signature of a given normed space structure, as well as the semantics of *approximate satisfaction*. These are the fundamental concepts on which the rest of our model theory is built.

A key aspect of approximate satisfaction is its well-behaved topological nature. This is illustrated by the Perturbation Lemma (Proposition 5.15) which expresses a sense in which satisfaction of positive bounded formulas is itself uniformly continuous on bounded sets. There are many important consequences of this, including the fact that the completion of any normed space structure is an approximate elementary extension of it.

6. Basic Model Theory I

The key model theoretic concepts of *approximate elementary equivalence* and *approximate elementary extension* are defined and their most basic properties are proved.

7. Quantifier-Free Formulas

The expressive power of quantifier-free positive bounded formulas in the pure language of normed spaces (with no additional structure) is explored.

8. Ultraproducts of Normed Space Structures

The ultraproduct of a family of normed space structures $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ with respect to any ultrafilter D on Λ is defined. Note that this cannot be defined in complete generality; even in the simple case where each $\mathcal{M}_\xi$ consists of a normed space X (not depending on the index ξ) together with a distinguished element x_ξ of X, the ultraproduct can be naturally defined for every D if and only if there is a real number r such that the bound $\|x_\xi\| \leq r$ holds for all $\xi \in \Lambda$. Similar considerations apply to boundedness and uniform continuity of the distinguished functions in the structures $\mathcal{M}_\xi$. We define what it means for a family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ to be *uniform* and note that this is precisely what is needed to be able to define the ultraproduct of the family. The most basic classes of normed space structures have the property that they are *uniform*, in the sense that any family taken from the class is uniform. This is true of normed spaces, normed lattices, and normed algebras, as is familiar. For expansions of such structures the requirement of uniformity is generally natural and easy to verify.

9. Basic Model Theory II

The fundamental theorem of ultraproducts is proved for the ultraproduct construction and the model theory that are being considered here. This is the analogue of the well known Theorem of Łoś from ordinary model theory. In this context it says the following: suppose $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a uniform family of normed space structures of the same signature, and let $\mathcal{N}$ be the Banach space ultraproduct of this family with respect to an ultrafilter D on Λ. Let σ be any positive bounded sentence in this signature. Then σ is approximately satisfied by $\mathcal{N}$ if and only if for each approximation σ' of σ, the set $\{\xi \in \Lambda \mid \sigma' \text{ is satisfied by } \mathcal{M}_\xi\}$ is an element of the ultrafilter D. The same equivalences are true for positive bounded formulas applied to elements of $\mathcal{N}$.

An immediate consequence of this theorem about ultraproducts is the Compactness Theorem for the model theory presented in this paper. Löwenheim-Skolem theorems and saturated structures are also considered in this chapter.

10. Isomorphic Ultrapowers

This chapter treats the analogue for this model theory of the Keisler-Shelah Theorem [Kei61] [She71] on isomorphism of ultrapowers and ultraproducts. Among other things, this yields the existence of ultrapowers of normed space structures that are κ-saturated, for any cardinal number κ.

11. Alternative Formulations of the Theory

The isomorphism results in the previous chapter are used to show that the expressive power of this model theory is not affected by various natural changes in how the sort of the real numbers is handled or in how the syntax of positive bounded formulas is defined.

12. Homogeneous Structures

The existence of ultrapowers that are κ-saturated and strongly κ-homogeneous is proved, using the theorems about isomorphic ultraproducts from an earlier chapter. This provides a family of richly endowed approximate elementary extensions of a given normed space structure, and these structures play an important role in applications of model theory.

13. More Model Theory

Let $\mathfrak{D}$ be a uniform class of structures with a common signature, and consider the class $\mathfrak{C}$ of structures for that signature that can be embedded in some ultraproduct of members of $\mathfrak{D}$. It is shown that $\mathfrak{C}$ consists of precisely the normed space structures for that signature that satisfy a certain set of positive bounded sentences in which only universal quantifiers appear.

A model theoretic characterization of axiomatizability for classes of normed space structures with a common signature is also discussed here. Let $\mathfrak{D}$ be a uniform class of structures with a common signature, and suppose that $\mathfrak{D}$ is closed under isomorphisms and ultraproducts. Let $\mathfrak{C}$ be contained in $\mathfrak{D}$. We show that $\mathfrak{C}$ is axiomatizable within $\mathfrak{D}$ by positive bounded sentences if and only if $\mathfrak{C}$ is closed under isomorphisms and ultraproducts and $\mathfrak{D} \setminus \mathfrak{C}$ is closed under ultrapowers (*i.e.* $\mathfrak{C}$ is closed under "ultraroots" within $\mathfrak{D}$).

This chapter also introduces a natural concept of *quantifier elimination* for normed space structures, and gives a model theoretic criterion for it. This concept often provides an effective basis for understanding what is definable by positive bounded formulas in a given structure or class of structures.

A more general result concerning the approximation of arbitrary positive bounded formulas by simpler formulas is also given here.

For many classes of structures from functional analysis, the structure of ultraproducts has been extensively investigated. When this work is combined with the results presented in this chapter, it is often possible to prove axiomatizability and even sometimes to prove quantifier elimination for interesting classes of such structures. In several cases this perspective raises interesting new questions about classes of normed space structures that have been previously studied; especially, it emphasizes the importance of understanding whether these classes are closed under "ultraroots" and whether they satisfy the mapping extension criterion for quantifier elimination.

14. Types

Spaces of types are a characteristic feature of modern applications of model theory in all areas of mathematics. Quantifier-free types have been used significantly in the geometry of Banach spaces, especially in connection with the class of stable spaces introduced (with direct motivation from work in model theory) by Krivine and Maurey [KM81]. Some fundamental aspects of this topic are developed in this chapter.

2

Normed Space Structures

2.1 Definition. A *normed space structure* $\mathcal{M}$ consists of the following items:

(i) A family $(M^{(s)} \mid s \in S)$ of normed spaces.
(ii) A collection of functions of the form

$$F\colon M^{(s_1)} \times \cdots \times M^{(s_m)} \to M^{(s_0)},$$

each of which is uniformly continuous (and hence bounded) on every bounded subset of its domain.

The normed spaces $M^{(s)}$ are called the *sorts* of $\mathcal{M}$, and we say that $\mathcal{M}$ is *based on* $(M^{(s)} \mid s \in S)$. If every sort of $\mathcal{M}$ is a Banach space (*i.e.*, is complete in its norm), we say that $\mathcal{M}$ is a *Banach space structure*.

Some of the functions of $\mathcal{M}$ may have arity 0; they correspond to distinguished elements of the sorts of $\mathcal{M}$. These elements are called the *constants* of $\mathcal{M}$. The other functions may be referred to as the *operations* of $\mathcal{M}$.

We require that every normed space structure include a special sort and certain special functions; we will refer to these functions as the *required operations* of the structure. Namely, if $\mathcal{M}$ is a normed space structure based on $(M^{(s)} \mid s \in S)$, then:

- The field $\mathbb{R}$ must occur as a sort of $\mathcal{M}$; *i.e.*, the sort index set S contains a distinguished element $s = s_{\mathbb{R}}$ for which $M^{(s)}$ is $\mathbb{R}$.
- For every $s \in S$, the vector space operations and the norm of $M^{(s)}$ must occur as functions of $\mathcal{M}$, and the additive identity $0_{M^{(s)}}$ of $M^{(s)}$ must occur as a constant of $\mathcal{M}$. (By *vector space operations* we mean $+$ and $-$ as binary functions from $M^{(s)} \times M^{(s)}$ into $M^{(s)}$ as well as scalar multiplication as a binary function from $\mathbb{R} \times M^{(s)}$ into $M^{(s)}$.)

· The distinguished operations on $\mathbb{R} = M^{(s_\mathbb{R})}$ consist of the usual field operations $+$, $-$, and $\times$, as well as the absolute value function (which is the norm on this sort). In addition to the additive identity 0 we require that each rational number be a distinguished constant in this sort.

If $\mathcal{M}$ is based on $(M^{(s)} \mid s \in S)$ and $(F_i \mid i \in I)$ is a list of the functions of $\mathcal{M}$, we may write

$$\mathcal{M} = (M^{(s)}, F_i \mid s \in S, i \in I).$$

The special sort $\mathbb{R}$ as well as the functions and constants which are required to be present in every normed space structure (see above) need not be listed explicitly in the preceding notation.

If $\mathcal{M}$ is based on $(M^{(s)} \mid s \in S)$ and $a \in M^{(s)}$ for some $s \in S$, we refer to a as an *element* of $\mathcal{M}$. Note that we make no assumptions about how the sorts of $\mathcal{M}$ are related to each other as sets. In particular $a \in M^{(s)}$ may be true for several different values of s. This means that when dealing with an element a of a normed space structure $\mathcal{M}$, it is necessary to specify the sort of which a is taken to be an element.

2.2 Examples. We give a number of examples of normed space structures to indicate the wide range of possibilities.

(i) Normed spaces X over $\mathbb{R}$: the sorts are X and $\mathbb{R}$, and the functions are the vector space operations, the additive identity 0_X and the norm of X, as well as the field operations, the additive identity 0 and the absolute value function on $\mathbb{R}$.

(ii) Normed spaces X over $\mathbb{C}$: these can be regarded as normed space structures in several ways. For example we may add $\mathbb{C}$ as a sort together with its field structure and absolute value, and the scalar multiplication operation as a map from $\mathbb{C} \times \mathbb{X}$ into X, as well as the inclusion map from $\mathbb{R}$ into $\mathbb{C}$. Alternatively, we may simply include a unary function from X into itself, corresponding to scalar multiplication by $\sqrt{-1}$, in addition to the usual operations that come from regarding X as a normed space over $\mathbb{R}$.

(iii) Normed vector lattices $(X, \vee, \wedge)$: this is the result of expanding the normed space structure corresponding to X (see above) by adding the lattice operations $\vee$ and $\wedge$ on X and the functions max and min on $\mathbb{R}$.

(iv) Normed algebras: multiplication is included as an operation; if the algebra has a multiplicative identity, it may be included as a constant.

(v) C^*-algebras: multiplication and the *-map are included as operations.

(vi) Hilbert spaces with inner product, in which the pairing is included as a function.

(vii) Dual pairs (X, X'), where X is a Banach space, X' is the dual of X, and the pairing between X and X' is included as a function.

(viii) Triples (X, X', X''), where X' and X'' are the dual and the double dual of X and the pairing between X and X', the pairing between X' and X'', and the embedding $X \to X''$ are included as functions.

(ix) Operator spaces, which include for each $n \geq 1$ a real-valued function of n^2 arguments mapping each $n \times n$ matrix (a_{ij}) of elements of the underlying Banach space to its operator norm.

(x) If $\mathcal{M}$ is a normed space structure and a is an element of $\mathcal{M}$, then the expansion $(\mathcal{M}, a)$ is a normed space structure, in which a is a constant.

(xi) If $\mathcal{M}$ is a normed space structure, and T is a bounded linear operator between sorts of $\mathcal{M}$, then the expansion $(\mathcal{M}, T)$ is a normed space structure, in which T is a distinguished function.

(xii) If $\mathcal{M}$ is a normed space structure, $M^{(s)}$ is a sort of $\mathcal{M}$, and A is a given subset of $M^{(s)}$, then $\mathcal{M}$ can be expanded by adding the real-valued function $x \mapsto \operatorname{dist}(x, A)$, where x ranges over $M^{(s)}$ and dist denotes the distance function with respect to the norm on $M^{(s)}$. The same can be done with subsets of finite cartesian products of sorts.

3
Signatures

In order to treat normed space structures $\mathcal{M}$ in general, it is convenient to introduce formally a way of indexing the sorts and functions of $\mathcal{M}$; this is the *signature* of $\mathcal{M}$.

3.1 Definition. Let $\mathcal{M}$ be a normed space structure based on $(M^{(s)} \mid s \in S)$. A *signature* L for $\mathcal{M}$ consists of the following items:

· A sort index set S.
· An element $s_{\mathbb{R}} \in S$ such that $M^{(s_{\mathbb{R}})} = \mathbb{R}$.
· For each designated function

$$F \colon M^{(s_1)} \times \cdots \times M^{(s_m)} \to M^{(s_0)}$$

of $\mathcal{M}$, a triple of the form

$$\big(f, (s_1, \ldots, s_m), s_0\big),$$

where f is a syntactic symbol called a *function symbol* for F. In this case, we write $F = f^{\mathcal{M}}$ and call F the *interpretation* of f in $\mathcal{M}$. We call $s_1 \times \cdots \times s_m$ the *domain* of f and s_0 the *range* of f, and express this by writing (purely formally)

$$f \colon s_1 \times \cdots \times s_m \to s_0.$$

The number m is called the *arity* of f. If $m = 0$, we call f a *constant symbol* and write $f : s_0$. In this case $f^{\mathcal{M}}$ is an element of $M^{(s_0)}$.

Each of the required functions of a normed space structure has a specific designated function symbol, and these are taken to be the same for each signature. For example, the additive identity of a sort $M^{(s)}$ is associated to a constant symbol designated $0 : s$, the vector addition and subtraction on this sort correspond to function symbols whose associated triples are $(+, (s, s), s)$ and $(-, (s, s), s)$, and so forth.

We express the fact that L is a signature for $\mathcal{M}$ by saying that $\mathcal{M}$ is an *L-structure.*

3.2 Definition. If L, L' are signatures, we write $L \subseteq L'$ if the following two conditions hold:

- The sort index set of L is a subset of the sort index set of L'.
- Every triple of the form $\big(f, (s_1, \ldots, s_m), s_0\big)$ which is in L is also in L'.

In this case we say L' is an *extension* of L and that L is a *subsignature* of L'.

Informally, $L \subseteq L'$ means that every sort and function symbol of L is also in L', and that they are used in precisely the same way in L' as they are in L.

If L, L' are signatures and $L \subseteq L'$, we call L' an *extension by constants* of L if L and L' have the same sort index set and if every function symbol of L' that is not in L is a constant symbol. In this situation we write $L' = L(C)$ where C is the set of constant symbols that have been added to L to get L'.

3.3 Definition. If L is a signature, the *cardinality of* L is the cardinal number $\mathrm{card}(L)$ defined to be equal to

$$\mathrm{card}(S) + \mathrm{card}(\{\, f \mid f \text{ a function symbol of } L \,\}) + \aleph_0.$$

In particular, we say L is *countable* if it has a finite or countable number of sorts and distinguished function symbols.

3.4 Definition. Let L be a signature and suppose that $\mathcal{M}$ and $\mathcal{N}$ are normed space L-structures based on $(M^{(s)} \mid s \in S)$ and $(N^{(s)} \mid s \in S)$, respectively. We say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ (or that $\mathcal{N}$ is an *extension* of $\mathcal{M}$) if $M^{(s)} \subseteq N^{(s)}$ for every $s \in S$ and $f^{\mathcal{N}}$ extends $f^{\mathcal{M}}$, for every function symbol f of L. We write $\mathcal{M} \subseteq \mathcal{N}$ to indicate that $\mathcal{M}$ is a substructure of $\mathcal{N}$.

Let $\mathcal{N}$ be as above and let $(A^{(s)} \mid s \in S)$ be a family of sets such that $A^{(s)} \subseteq N^{(s)}$ for every $s \in S$. The substructure of $\mathcal{N}$ *generated by* $(A^{(s)} \mid s \in S)$ is the L-substructure $\mathcal{M}$ of $\mathcal{N}$ that results from closing $(A^{(s)} \mid s \in S)$ and $\mathbb{R} = M^{(s_\mathbb{R})}$ under all the functions of $\mathcal{M}$. We say that $\mathcal{N}$ is *finitely generated* if $\mathcal{N}$ is generated by a family $(A^{(s)} \mid s \in S)$ such that $A^{(s)}$ is finite for every $s \in S$ and $A^{(s)} = \emptyset$ for all but finitely many $s \in S$.

Informally we may say that $\mathcal{M} \subseteq \mathcal{N}$ holds if and only if $M^{(s)}$ is a normed subspace of $N^{(s)}$ for every $s \in S$ and $f^{\mathcal{N}}$ maps $\mathcal{M}$ into itself, for every function symbol f of L.

3.5 Definition. Let L be a signature and suppose that $\mathcal{M}$ and $\mathcal{N}$ are normed space L-structures based on $(M^{(s)} \mid s \in S)$ and $(N^{(s)} \mid s \in S)$, respectively. An *isomorphism* between $\mathcal{M}$ and $\mathcal{N}$ consists of a collection of surjective isometries $T^{(s)}\colon M^{(s)} \to N^{(s)}$, for $s \in S$, which commute with the interpretations of the function symbols of L in the following sense: whenever $f\colon s_1 \times \cdots \times s_m \to s_0$ is a function symbol of L and $a_k \in M^{(s_k)}$ for $k = 1, \ldots, m$, we have

$$f^{\mathcal{N}}(\,T^{(s_1)}(a_1), \ldots, T^{(s_m)}(a_m)\,) = T^{(s_0)}(\,f^{\mathcal{M}}(a_1, \ldots, a_m)\,).$$

We say that $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic*, and write $\mathcal{M} \cong \mathcal{N}$, if there exists an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. (Sometimes we say *isometric isomorphism* to emphasize that isomorphisms must be norm preserving; since the norm on each sort is included as a distinguished function in every normed space structure, it must be preserved by every isomorphism.)

An *embedding* (equivalently, *isometric embedding*) of $\mathcal{M}$ into $\mathcal{N}$ is an isomorphism between $\mathcal{M}$ and a substructure of $\mathcal{N}$.

An *automorphism* (equivalently, *isometric automorphism*) of $\mathcal{M}$ is an isomorphism between $\mathcal{M}$ and $\mathcal{M}$.

Note that if $\mathcal{M}$ and $\mathcal{N}$ are normed space L-structures and $(T^{(s)} \mid s \in S)$ is an embedding of $\mathcal{M}$ into $\mathcal{N}$, then $T^{(s_{\mathbb{R}})}$ must be the identity map of $M^{(s_{\mathbb{R}})}$ to $N^{(s_{\mathbb{R}})}$.

3.6 Definition. Let L be a signature and suppose that $\mathcal{M}$ is an L-structure based on $(M^{(s)} \mid s \in S)$.

(i) The *completion* of $\mathcal{M}$ is the Banach space L-structure $\overline{\mathcal{M}}$ defined as follows. For each $s \in S$ let $\overline{M^{(s)}}$ denote the norm completion of $M^{(s)}$. Then $\overline{\mathcal{M}}$ is based on $(\overline{M^{(s)}} \mid s \in S)$, and for every function symbol f of L, $f^{\overline{\mathcal{M}}}$ is defined to be the unique continuous extension of $f^{\mathcal{M}}$. (The domain D of $f^{\overline{\mathcal{M}}}$ is determined since $\overline{\mathcal{M}}$ is an L-structure based on a specific family of sorts, and D has the domain of $f^{\mathcal{M}}$ as a dense subspace.)

(ii) If $\mathcal{M} \subseteq \mathcal{N}$, the *closure* of $\mathcal{M}$ in $\mathcal{N}$ is the substructure $\mathcal{M}'$ of $\mathcal{N}$ defined as follows. For each $s \in S$ let $(M^{(s)})'$ denote the closure of $M^{(s)}$ in $\mathcal{N}$. Then $\mathcal{M}'$ is the substructure of $\mathcal{N}$ that is based on $((M^{(s)})' \mid s \in S)$. (Continuity of the functions of $\mathcal{N}$ ensures that there is such a substructure.)

Evidently each normed space structure is a substructure of its completion and of its closure in any extension.

3.7 Definition. Let L, L' be signatures with $L \subseteq L'$ and suppose that $\mathcal{N}$ is an L'-structure. The *reduct* of $\mathcal{N}$ to L is the L-structure that results by removing from $\mathcal{N}$ the sorts and functions that are indexed by L' but not by L. We say that a normed space $\mathcal{N}$ is an *expansion* of a structure $\mathcal{M}$ if $\mathcal{M}$ is a reduct of $\mathcal{N}$.

We say that a normed space L'-structure $\mathcal{N}$ is an *expansion by constants* of an L-structure $\mathcal{M}$ if L' is an extension by constants of L and $\mathcal{N}$ is an expansion of $\mathcal{M}$.

Suppose $L(C)$ is an extension by constants of L, $\mathcal{M}$ is a normed space L-structure based on $(M^{(s)} \mid s \in S)$, and the $L(C)$-structure $\mathcal{N}$ is an expansion of $\mathcal{M}$. In this situation we denote $\mathcal{N}$ by $(\mathcal{M}, \mathbf{A})$, where $\mathbf{A}$ is the function defined on C by setting $\mathbf{A}(c) = c^{\mathcal{N}}$ for each $c \in C$. Suppose $\mathcal{M}$ is based on $(M^{(s)} \mid s \in S)$. Note that $\mathbf{A}$ may be any function from C to elements of $\mathcal{M}$ which satisfies the requirement that for each $c \in C$, if the range of c is s, then $\mathbf{A}(c) \in M^{(s)}$. We refer to the function $\mathbf{A}$ as an *interpretation of C in* $\mathcal{M}$.

4
Ultrapowers of Normed Space Structures

Ultrafilter Limits

Let X be a topological space. Further, let $(x_\xi)_{\xi\in\Lambda}$ be an indexed family in X. If D is an ultrafilter on Λ and $x \in X$, we write

$$\lim_{\xi,D} x_\xi = x$$

and say that x is the *D-limit* of $(x_\xi)_{\xi\in\Lambda}$ if for every neighborhood U of x, the set $\{\, \xi \in \Lambda \mid x_\xi \in U \,\}$ is in the ultrafilter D. A basic fact from general topology is that X is a compact Hausdorff space if and only if for every indexed family $(x_\xi)_\xi \in \Lambda$ in X and every ultrafilter D on Λ the D-limit of $(x_\xi)_{\xi\in\Lambda}$ exists and is unique.

Ultraproducts of Normed Spaces

Let $(X_\xi \mid \xi \in \Lambda)$ be a family of normed spaces. Define

$$\ell_\infty\big(\Lambda, (X_\xi \mid \xi \in \Lambda)\big) = \{\ (x_\xi)_{\xi\in\Lambda} \in \prod_{\xi\in\Lambda} X_\xi \ \mid\ \sup_{\xi\in\Lambda} \|x_\xi\| < \infty\ \}.$$

One sees that $\ell_\infty(\Lambda, (X_\xi \mid \xi \in \Lambda))$ is naturally a normed space with respect to the norm defined by

$$\|(x_\xi)_{\xi\in\Lambda}\|_\infty = \sup_{\xi\in\Lambda} \|x_\xi\|.$$

An ultrafilter D on Λ induces a continuous seminorm on $\ell_\infty(\Lambda, (X_\xi \mid \xi \in \Lambda))$ by defining

$$\|(x_\xi)_{\xi\in\Lambda}\| = \lim_{\xi,D} \|x_\xi\|$$

for each $(x_\xi)_{\xi\in\Lambda} \in \ell_\infty(\Lambda, (X_\xi \mid \xi \in \Lambda))$. The set N_D of families $(x_\xi)_{\xi\in\Lambda}$ in $\ell_\infty(\Lambda, (X_\xi \mid \xi \in \Lambda))$ such that $\|(x_\xi)_{\xi\in\Lambda}\| = 0$ is a closed subspace of

$\ell_\infty(\Lambda, (X_\xi \mid \xi \in \Lambda))$. We define

$$\Big(\prod_{\xi\in\Lambda} X_\xi\Big)_D = \Big(\ell_\infty(\Lambda, (X_\xi \mid \xi \in \Lambda))\Big)\Big/ N_D.$$

The space $(\prod_{\xi\in\Lambda} X_\xi)_D$ is called the *D-ultraproduct of* $(X_\xi \mid \xi \in \Lambda)$. If $(x_\xi)_{\xi\in\Lambda} \in \ell_\infty(\Lambda, (X_\xi \mid \xi \in \Lambda))$ we denote the equivalence class of $(x_\xi)_{\xi\in\Lambda}$ in $(\prod_{\xi\in\Lambda} X_\xi)_D$ by $((x_\xi)_{\xi\in\Lambda})_D$.

If $X_\xi = X$ for every $\xi \in \Lambda$, the space $(\prod_{\xi\in\Lambda} X_\xi)_D$ is called the *D-ultrapower of* X and is denoted $(X)_D$.

If X is a normed space and D is an ultrafilter on Λ, then the map $T\colon X \to (X)_D$ defined by $T(x) = ((x_\xi)_{\xi\in\Lambda})_D$, where $x_\xi = x$ for every $\xi \in \Lambda$, is an isometric embedding. It is called the *diagonal embedding* of X into $(X)_D$. This embedding is generally not surjective; it is, however, when the ultrafilter D is principal or the space X is finite dimensional.

Ultrapowers of Functions

4.1 Notation. If X is a normed space and r is a nonnegative real number, we let $\mathcal{B}_r(X)$ denote the closed ball of radius r around 0 in X.

4.2 Definition. Suppose that $X^{(0)}, \dots, X^{(m)}$ are normed spaces and that

$$F : X^{(1)} \times \cdots \times X^{(m)} \to X^{(0)}$$

is uniformly continuous on a subset B of $X^{(1)} \times \cdots \times X^{(m)}$. A *modulus of uniform continuity for F on B* is a map $\Delta\colon (0,\infty) \to (0,\infty)$ such that for any $(x_1, \dots, x_m), (y_1, \dots, y_m) \in B$ and any $\epsilon > 0$,

$$\max_k \|x_k - y_k\| < \Delta(\epsilon) \quad \text{implies} \quad \|F(x_1, \dots, x_m) - F(y_1, \dots, y_m)\| \le \epsilon.$$

Suppose that $X^{(0)}, \dots, X^{(m)}$ are normed spaces and that

$$F : X^{(1)} \times \cdots \times X^{(m)} \to X^{(0)}$$

is uniformly continuous on every bounded subset of its domain. Given an ultrafilter D on Λ, we define a function

$$(F)_D\colon (X^{(1)})_D \times \cdots \times (X^{(m)})_D \to (X^{(0)})_D$$

as follows: if $(x^k_\xi)_{\xi\in\Lambda} \in \ell_\infty(\Lambda, (X^{(k)})^\Lambda)$ for $k = 1, \dots, m$, then

$$(F)_D\big(((x^1_\xi)_{\xi\in\Lambda})_D, \dots, ((x^m_\xi)_{\xi\in\Lambda})_D\big) = \big(\big(F(x^1_\xi, \dots, x^m_\xi)\big)_{\xi\in\Lambda}\big)_D.$$

The uniform continuity assumption on F ensures that $(F)_D$ is well-defined; in fact, we have the following:

4.3 Proposition. *Let F and $(F)_D$ be as above. Any modulus of uniform continuity for F on $\mathcal{B}_{r_1}(X^{(1)}) \times \cdots \times \mathcal{B}_{r_m}(X^{(m)})$ is also a modulus of uniform continuity for $(F)_D$ on $\mathcal{B}_{r_1}((X^{(1)})_D) \times \cdots \times \mathcal{B}_{r_m}((X^{(m)})_D)$.*

Proof. This is a consequence of the definition of D-limit. $\square$

The fact that $(F)_D$ is well-defined depends in part on the observation that if B is a bounded convex subset of the domain of F and F is uniformly continuous on B, then F is uniformly bounded on B. Indeed, if the diameter of B is $\leq d$, x is any element of B, and Δ is any modulus of uniform continuity for F on B, then for any $y \in B$ we have

$$\|F(y)\| \leq \|F(x)\| + d \cdot \Delta(1)^{-1}.$$

Ultrapowers of Normed Space Structures

4.4 Definition. Let L be a signature and suppose that $\mathcal{M}$ is a normed space L-structure based on $(M^{(s)} \mid s \in S)$. If D is an ultrafilter, the *D-ultrapower* of $\mathcal{M}$ is the normed space L-structure defined as follows. The sorts of $(\mathcal{M})_D$ are the D-ultrapowers $(M^{(s)})_D$, for $s \in S$, and for every function symbol f of L, one sets $f^{(\mathcal{M})_D} = (f^{\mathcal{M}})_D$.

4.5 Remark. If $\mathcal{M}$ is a normed space structure based on $(M^{(s)} \mid s \in S)$ and $T^{(s)}\colon M^{(s)} \to (M^{(s)})_D$ is the diagonal embedding for each $s \in S$, then the family $(T^{(s)} \mid s \in S)$ is an embedding of $\mathcal{M}$ into $(\mathcal{M})_D$. We call this family the *diagonal embedding* of $\mathcal{M}$ into $(\mathcal{M})_D$.

Nonstandard Hulls

Let $\mathcal{M}$ be a normed space L-structure and let D be an ultrafilter on the index set Λ. The ultrapower $(\mathcal{M})_D$ is an example of a *nonstandard hull* of $\mathcal{M}$. This more general construction is widely used in applications of nonstandard analysis. See [HL85] [Lin88] [Hen97] for background on this construction and the general area of nonstandard analysis.

Given a set theoretic superstructure V of which $\mathcal{M}$ is an element, let ${}^*V$ be the D-ultrapower nonstandard extension of V constructed using the ultrafilter D. (See, for example, Section II.4 in [HL85] or the proof of Theorem 2.28 in [Hen97].) The $*$-mapping from V into ${}^*V$ takes $\mathcal{M}$ to the internal object ${}^*\mathcal{M}$; this object is represented by the constant family $(\mathcal{M} \mid \xi \in \Lambda)$.

In general, when ${}^*V$ is any nonstandard extension of V, the nonstandard hull is constructed from ${}^*\mathcal{M}$ by (a) keeping *finite* elements of

${}^*\mathcal{M}$ (elements whose internal norm is a finite element of ${}^*\mathbb{R}$) and (b) identifying two finite elements x, y of the same sort of ${}^*\mathcal{M}$ if $x - y$ is *infinitesimal* (*i.e.* its internal norm is an infinitesimal element of ${}^*\mathbb{R}$). The uniform continuity requirement on the operations F of $\mathcal{M}$ ensures that ${}^*F$ maps finite arguments to finite values and preserves infinitesimal equivalence, so it gives rise to a well-defined quotient operation. The resulting structure is a normed space L-structure, denoted by $\widehat{\mathcal{M}}$.

Again consider the situation where ${}^*V$ is the D-ultrapower nonstandard extension of V. It is easy to check that finite elements of ${}^*\mathbb{R}$ have representatives which are bounded families of real numbers; likewise, infinitesimal elements of ${}^*\mathbb{R}$ are represented by families of real numbers whose D-limit is 0. It follows that the nonstandard hull $\widehat{\mathcal{M}}$ constructed using this ${}^*V$ is isomorphic, as a normed space L-structure, to $(\mathcal{M})_D$.

5
Positive Bounded Formulas

In this chapter we introduce the key ingredients of the logic for normed space structures that is described in this paper. These are the *positive bounded formulas* and the concept of *approximate satisfaction* of such formulas in normed space structures.

Let L be a signature for a normed space structure $\mathcal{M}$ based on $(M^{(s)} \mid s \in S)$. Recall that S has a distinguished element $s = s_{\mathbb{R}}$ for which $M^{(s)} = \mathbb{R}$ is the sort of real numbers.

We begin considering $\mathcal{M}$ from the model theoretic point of view, introducing a formal language based on L and a semantics according to which this language is interpreted in $\mathcal{M}$. In addition to the symbols of the signature L, we also need for each element s of the sort index set S, a countable set of symbols called the *variables of sort s*.

We begin defining the formal language by introducing the set of *terms of L*, or *L-terms*. Each term is a finite string of symbols, each of which may be a variable or a function symbol of L, or one of the symbols (or , which are used for punctuation. In this many-sorted context, each term is associated with a unique sort which indicates its range. The formal definition is recursive.

5.1 Definition. An *L-term with range of sort s* is a string which can be obtained by finitely many applications of the following rules of formation:

- If x is a variable of L of sort s, then x is a term with range of sort s.
- If f is a function symbol of L with $f\colon s_1 \times \cdots \times s_m \to s_0$ and t_k is a term with range of sort s_k for $k = 1, \ldots, m$, then $f(t_1, \ldots, t_m)$ is a term with range of sort s_0.

We say that t is a *term* if t is a term with range of sort s for some $s \in S$; further, t is a *real-valued term* if t is a term with range of sort $s_{\mathbb{R}}$.

5.2 Definition. Let L be a signature for normed space structures. The set of *positive bounded L-formulas* is defined recursively as follows.

- If t is a real-valued term and r is a fixed rational number, then the expressions
$$r \leq t, \qquad t \leq r$$
are positive bounded formulas.
- If φ_1 and φ_2 are positive bounded formulas, then the expressions
$$(\varphi_1 \wedge \varphi_2), \qquad (\varphi_1 \vee \varphi_2)$$
are positive bounded formulas.
- If φ is a positive bounded formula, x is a variable, and r is a positive rational number, then the expressions
$$\exists x(\|x\| \leq r \wedge \varphi),$$
$$\forall x(\|x\| \leq r \rightarrow \varphi)$$
are positive bounded formulas.

Thus, the positive bounded formulas are built from the *atomic formulas*
$$r \leq t, \qquad t \leq r$$
by finite application of the *positive connectives* $\wedge$, $\vee$ and the *bounded quantifiers*
$$\exists x(\|x\| \leq r \wedge \dots),$$
$$\forall x(\|x\| \leq r \rightarrow \dots)$$

If t is a term and r_1, r_2 are real numbers, we write $r_1 \leq t \leq r_2$ as an abbreviation of the positive bounded formula $(r_1 \leq t \wedge t \leq r_2)$. Similarly, we write $t = r$ as an abbreviation for $r \leq t \wedge t \leq r$. When the context allows it, we often omit the outer parentheses in formulas of the form $(\varphi_1 \wedge \varphi_2)$ or $(\varphi_1 \vee \varphi_2)$. Sometimes we also write $\bigwedge_{i=1}^{n} \varphi_i$ and $\bigvee_{i=1}^{n} \varphi_i$ as abbreviations of $\varphi_1 \wedge \cdots \wedge \varphi_n$ and $\varphi_1 \vee \cdots \vee \varphi_n$, respectively.

The bounded quantifiers may also be abbreviated. We write $\exists_r x\, \varphi$ and $\forall_r x\, \varphi$ as abbreviations of $\exists x(\|x\| \leq r \wedge \varphi)$ and $\forall x(\|x\| \leq r \rightarrow \varphi)$, respectively.

Let φ be a positive bounded formula. A *subformula* of φ is any string of consecutive symbols of φ which is a positive bounded formula in its own right. A variable x is said to be *free* in a positive bounded formula

φ if there is at least one occurrence of x in φ which is not within any subformula of φ of the form $\exists_r x\, \psi$ or $\forall_r x\, \psi$. A *positive bounded sentence* is a positive bounded formula without free variables.

If t is a term and $x_1, \ldots, x_n$ are distinct variables, we will write $t(x_1, \ldots, x_n)$ to indicate that all variables occurring in t are among $x_1, \ldots, x_n$. Similarly, if φ is a positive bounded formula, we write $\varphi(x_1, \ldots, x_n)$ to indicate that all variables occurring free in φ are among $x_1, \ldots, x_n$. Also, $\Gamma(x_1, \ldots, x_n)$ denotes a set of positive bounded formulas whose free variables are among $x_1, \ldots, x_n$.

Each term t defines a function $t^{\mathcal{M}}$ on $\mathcal{M}$, as specified in the following definition:

5.3 Definition. Suppose that $\mathcal{M}$ is a normed space L-structure based on $(M^{(s)} \mid s \in S)$. Let $t(x_1, \ldots, x_n)$ be an L-term, where x_k is a variable of sort s_k for $k = 1, \ldots, n$, and let $a_1, \ldots, a_n$ be such that $a_k \in M^{(s_k)}$ for $k = 1, \ldots, n$. We define the *evaluation of* t *in* $\mathcal{M}$ *at* $a_1, \ldots a_n$, denoted

$$t^{\mathcal{M}}[a_1, \ldots, a_n],$$

as follows. The definition is by induction on the complexity of t. It ensures that $t^{\mathcal{M}}[a_1, \ldots, a_n] \in M^{(s)}$ holds whenever t is a term with range of sort s.

- If t is x_k, then $t^{\mathcal{M}}[a_1, \ldots, a_n]$ is a_k;
- If t is $f(t_1, \ldots, t_m)$, where f is a function symbol of L and $t_1, \ldots, t_m$ are terms, then $t^{\mathcal{M}}[a_1, \ldots, a_n]$ is

$$f^{\mathcal{M}}(t_1^{\mathcal{M}}[a_1, \ldots, a_n], \ldots, t_m^{\mathcal{M}}[a_1, \ldots, a_n]).$$

5.4 Remark. Note that $t^{\mathcal{M}}[a_1, \ldots, a_n]$ depends not only on t, $\mathcal{M}$ and $a_1, \ldots, a_n$, but also on a given list of variables which is not explicit in the notation. The context will make it clear which list of variables is being used.

5.5 Definition. Suppose that $\mathcal{M}$ is a normed space L-structure based on $(M^{(s)} \mid s \in S)$. Let $\varphi(x_1, \ldots, x_n)$ be a positive bounded L-formula, where x_k is a variable of sort s_k for $k = 1, \ldots, n$, and let $a_1, \ldots, a_n$ be such that $a_k \in M^{(s_k)}$ for $k = 1, \ldots, n$. We define the relation

$$\mathcal{M} \models \varphi[a_1, \ldots, a_n].$$

When this holds, we say that $\mathcal{M}$ *satisfies* $\varphi(x_1, \ldots, x_n)$ *at* $a_1, \ldots, a_n$, or that $\varphi(x_1, \ldots, x_n)$ is *true in* $\mathcal{M}$ *at* $a_1, \ldots, a_n$. The definition of $\mathcal{M} \models \varphi[a_1, \ldots, a_n]$ is by induction on the complexity of φ.

- If φ is $t(x_1,\ldots,x_n) \le r$, where t is a real-valued term and r is a rational number, then $\mathcal{M} \models \varphi[a_1,\ldots,a_n]$ if and only if

$$t^{\mathcal{M}}[a_1,\ldots,a_n] \le r.$$

- If φ is $r \le t(x_1,\ldots,x_n)$, where t is a real-valued term and r is a rational number, then $\mathcal{M} \models \varphi[a_1,\ldots,a_n]$ if and only if

$$r \le t^{\mathcal{M}}[a_1,\ldots a_n].$$

- If φ is $(\psi_1 \wedge \psi_2)$, then $\mathcal{M} \models \varphi[a_1,\ldots,a_n]$ if and only if

$$\mathcal{M} \models \psi_1[a_1,\ldots,a_n] \quad \text{and} \quad \mathcal{M} \models \psi_2[a_1,\ldots,a_n].$$

- If φ is $(\psi_1 \vee \psi_2)$, then $\mathcal{M} \models \varphi[a_1,\ldots,a_n]$ if and only if

$$\mathcal{M} \models \psi_1[a_1,\ldots,a_n] \quad \text{or} \quad \mathcal{M} \models \psi_2[a_1,\ldots,a_n].$$

- If φ is $\exists_r x\, \psi(x,x_1,\ldots,x_n)$, where r is a positive rational number and x is a variable of sort s, then $\mathcal{M} \models \varphi[a_1,\ldots,a_n]$ if and only if

$$\mathcal{M} \models \psi[a,a_1,\ldots,a_n], \quad \text{for some } a \in M^{(s)} \text{ with } \|a\| \le r.$$

- If φ is $\forall_r x\, \psi(x,x_1,\ldots,x_n)$, where r is a positive rational number and x is a variable of sort s, then $\mathcal{M} \models \varphi[a_1,\ldots,a_n]$ if and only if

$$\mathcal{M} \models \psi[a,a_1,\ldots,a_n], \quad \text{for every } a \in M^{(s)} \text{ with } \|a\| \le r.$$

5.6 Definition. If $\mathcal{M}$ is a normed space L-structure and $\Gamma(x_1,\ldots,x_n)$ is a set of positive bounded L-formulas, we write

$$\mathcal{M} \models \Gamma[a_1,\ldots,a_n]$$

if $\mathcal{M} \models \varphi[a_1,\ldots,a_n]$ for every $\varphi \in \Gamma$. In this case we say that $\mathcal{M}$ *satisfies* $\Gamma(x_1,\ldots,x_n)$ *at* $a_1,\ldots,a_n$, or that $\Gamma(x_1,\ldots,x_n)$ is *true in* $\mathcal{M}$ *at* $a_1,\ldots,a_n$.

Approximations

We now define the concept of *approximation* of a positive bounded formula. Intuitively, if φ is a positive bounded formula, an approximation φ' of φ is a positive bounded formula that results from relaxing all the norm estimates in φ. We indicate that φ' is an approximation of φ by writing $\varphi' > \varphi$ or equivalently $\varphi < \varphi'$. The formal definition is by induction on the complexity of φ:

If φ is:	The approximations of φ are:
$r \le t$	$r' \le t$, where $r' < r$
$t \le r$	$t \le r'$, where $r' > r$
$(\psi_1 \wedge \psi_2)$	$(\psi_1' \wedge \psi_2')$, where $\psi_i' > \psi_i$, for $i = 1, 2$
$(\psi_1 \vee \psi_2)$	$(\psi_1' \vee \psi_2')$, where $\psi_i' > \psi_i$, for $i = 1, 2$
$\exists_r x\, \psi$	$\exists_{r'} x\, \psi'$, where $r' > r$ and $\psi' > \psi$
$\forall_r x\, \psi$	$\forall_{r'} x\, \psi'$, where $r' < r$ and $\psi' > \psi$

5.7 Remark. Suppose that $\mathcal{M}$ is an L-structure and $\varphi(x_1, \dots, x_n)$ is a positive bounded formula. Then

$$\mathcal{M} \models \varphi[a_1, \dots, a_n] \quad \text{implies} \quad \mathcal{M} \models \varphi'[a_1, \dots, a_n], \quad \text{for every } \varphi' > \varphi.$$

5.8 Notation. If Γ is a set of positive bounded formulas, we let Γ^+ denote the set of approximations of formulas in Γ.

Approximate Satisfaction

5.9 Definition. Suppose that $\mathcal{M}$ is a normed space L-structure based on $(M^{(s)} \mid s \in S)$. Let $\varphi(x_1, \dots, x_n)$ be an L-formula, where x_k is a variable of sort s_k, for $k = 1, \dots, n$, and let $a_1, \dots, a_n$ be such that $a_k \in M^{(s_k)}$, for each k. We say that $\mathcal{M}$ *approximately satisfies* $\varphi(x_1, \dots, x_n)$ *at* $a_1, \dots, a_n$, or that $\varphi(x_1, \dots, x_n)$ is *approximately true* at $a_1, \dots, a_n$ in $\mathcal{M}$, and we write

$$\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \dots, a_n],$$

if $\mathcal{M} \models \varphi'[a_1, \dots, a_n]$ for every approximation φ' of φ.

5.10 Definition. If $\mathcal{M}$ is a normed space L-structure and $\Gamma(x_1, \dots, x_n)$ is a set of positive bounded L-formulas, we say that $\mathcal{M}$ *approximately satisfies* $\Gamma(x_1, \dots, x_n)$ *at* $a_1, \dots, a_n$ (or that $\Gamma(x_1, \dots, x_n)$ is *approximately true* in $\mathcal{M}$ at $a_1, \dots, a_n$), and write $\mathcal{M} \models_{\mathcal{A}} \Gamma[a_1, \dots, a_n]$, if $\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \dots, a_n]$ for every $\varphi \in \Gamma$.

5.11 Remark. Let Γ be as in the preceding definition and define Γ^+ as in 5.8. Then $\mathcal{M} \models_{\mathcal{A}} \Gamma[a_1, \dots, a_n]$ if and only if $\mathcal{M} \models \Gamma^+[a_1, \dots, a_n]$.

5.12 Notation. Suppose that $\mathcal{M}$ is a normed space L-structure based on $(M^{(s)} \mid s \in S)$ and let $\varphi(x_1, \dots, x_n)$ be an L-formula, where x_k is a variable of sort s_k, for $k = 1, \dots, n$. The relations $\mathcal{M} \models \varphi[a_1, \dots, a_n]$

and $\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]$ are defined only when $a_k \in M^{(s_k)}$ holds for $k = 1, \ldots, n$. Informally, we express this requirement by saying that $a_1, \ldots, a_n$ are elements *of suitable sorts of* $\mathcal{M}$.

Weak Negation

The negation connective is not allowed in positive bounded formulas, nor is the implication connective, except when it occurs as part of a bounded universal quantifier. However, for every positive bounded formula φ there is a positive bounded formula $\mathrm{neg}(\varphi)$ which in Banach space model theory plays a role analogous to that of the negation of φ. The operation neg is defined by induction on the complexity of formulas, as follows.

If φ is:	$\mathrm{neg}(\varphi)$ is:
$r \leq t$	$t \leq r$
$t \leq r$	$r \leq t$
$(\psi_1 \wedge \psi_2)$	$(\mathrm{neg}(\psi_1) \vee \mathrm{neg}(\psi_2))$
$(\psi_1 \vee \psi_2)$	$(\mathrm{neg}(\psi_1) \wedge \mathrm{neg}(\psi_2))$
$\exists_r x\, \psi$	$\forall_r x\ \mathrm{neg}(\psi)$
$\forall_r x\, \psi$	$\exists_r x\ \mathrm{neg}(\psi)$

5.13 Remarks. Suppose that $\varphi(x_1, \ldots, x_n)$,$\varphi'(x_1, \ldots, x_n)$ are positive bounded L-formulas. Then the following conditions hold whenever $\mathcal{M}$ is a normed space L-structure and $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$.

(i) $\mathrm{neg}(\mathrm{neg}(\varphi))$ is φ.
(ii) $\varphi < \varphi'$ if and only if $\mathrm{neg}(\varphi') < \mathrm{neg}(\varphi)$.
(iii) If $\mathcal{M} \not\models \varphi[a_1, \ldots, a_n]$, then $\mathcal{M} \models \mathrm{neg}(\varphi)[a_1, \ldots, a_n]$.
(iv) If $\varphi < \varphi'$ and $\mathcal{M} \models \mathrm{neg}(\varphi')[a_1, \ldots, a_n]$, then $\mathcal{M} \not\models \varphi[a_1, \ldots, a_n]$.

5.14 Proposition. *Suppose that $\mathcal{M}$ is a normed space L-structure. Let $\varphi(x_1, \ldots, x_n)$ be a positive bounded L-formula, and let $a_1, \ldots, a_n$ be elements of suitable sorts of $\mathcal{M}$. Then $\mathcal{M} \not\models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]$ if and only there exists $\varphi' > \varphi$ such that $\mathcal{M} \models_{\mathcal{A}} \mathrm{neg}(\varphi')[a_1, \ldots, a_n]$.*

Proof. If $\mathcal{M} \not\models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]$, there exists $\varphi' > \psi$ such that $\mathcal{M} \not\models \varphi'[a_1, \ldots, a_n]$. Then $\mathcal{M} \models \mathrm{neg}(\varphi')[a_1, \ldots, a_n]$ and hence

$$\mathcal{M} \models_{\mathcal{A}} \mathrm{neg}(\varphi')[a_1, \ldots, a_n]. \tag{*}$$

Assume, conversely, that there exists $\varphi' > \varphi$ such that (*) holds and take formulas ψ, ψ' such that $\varphi < \psi < \psi' < \varphi'$. By Remark 5.13,

$$\mathcal{M} \models \mathrm{neg}(\psi')[a_1, \ldots, a_n]$$

and hence $\mathcal{M} \not\models \psi[a_1, \ldots, a_n]$, so $\mathcal{M} \not\models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]$. □

Perturbation Lemma

If $\mathcal{M}$ is a normed space L-structure and t is an L-term, then $t^{\mathcal{M}}$ (see Definition 5.3) is a uniformly continuous function on any bounded subset of its domain. The next result shows that positive bounded formulas have a similar property.

This result first appeared, in the context of Banach spaces without additional structure, as [HM83a, Proposition 2.3]. It is the basis for many of the most important features of positive bounded formulas and approximate satisfaction.

5.15 Proposition (Perturbation Lemma). *Let a normed space L-structure $\mathcal{M}$ and a positive bounded L-formula $\varphi(x_1, \ldots, x_n)$ be given. Then for every $r > 0$ and every approximation φ' of φ there exists $\delta > 0$ such that whenever $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are elements of suitable sorts of $\mathcal{M}$ we have that*

$$\mathcal{M} \models \bigwedge_{1 \leq i \leq n} \|a_i\| \leq r \ \wedge \bigwedge_{1 \leq i \leq n} \|a_i - b_i\| \leq \delta \ \wedge \ \varphi[a_1, \ldots, a_n]$$

implies

$$\mathcal{M} \models \varphi'[b_1, \ldots, b_n].$$

Proof. By induction on the complexity of φ. □

5.16 Corollary. *Let $\varphi(x_1, \ldots, x_n)$ be a positive bounded L-formula and let $\mathcal{M}$ be a normed space L-structure. Then the set*

$$\varphi^{\mathcal{M}} = \{(a_1, \ldots, a_n) \mid \mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]\}$$

is norm-closed.

Proof. Suppose $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$ such that $(a_1, \ldots, a_n) \notin \varphi^{\mathcal{M}}$. By definition, there exists an approximation ψ' of φ such that

$$\mathcal{M} \not\models \psi'[a_1, \ldots, a_n].$$

Choose another approximation ψ such that $\varphi < \psi < \psi'$. Let r be a real number such that $\|a_j\| + 1 \leq r$ for all $j = 1, \ldots, n$. From the Perturbation Lemma obtain δ with $0 < \delta \leq 1$ for the formulas ψ, ψ' and the norm bound r. Whenever $b_1, \ldots, b_n$ are elements of suitable sorts with $\|b_j - a_j\| < \delta$ for all $j = 1, \ldots, n$ it follows that

$$\mathcal{M} \not\models \psi[b_1, \ldots, b_n]$$

and hence

$$\mathcal{M} \not\models_{\mathcal{A}} \varphi[b_1, \ldots, b_n],$$

so $(b_1, \ldots, b_n) \notin \varphi^{\mathcal{M}}$. This proves that $\varphi^{\mathcal{M}}$ is closed for the norm topology. $\square$

Satisfaction and Approximate Satisfaction

Suppose L is a signature and $\mathcal{M}$ is a normed space L-structure. If $\varphi(x_1, \ldots, x_n)$ is a positive bounded L-formula without quantifiers, it is easy to show that for any elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}$,

$$\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathcal{M} \models \varphi[a_1, \ldots, a_n].$$

It follows that the same equivalence holds when φ is a positive bounded L-formula in which every bounded quantifier is universal (*i.e.*, of the form $\forall_r x$).

However, for a general positive bounded formula φ, it is usually false that $\models_{\mathcal{A}}$ and $\models$ have the same meaning. The most general thing one can say (see Remark 5.7) is that for any normed space L-structure $\mathcal{M}$ and any elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}$,

$$\mathcal{M} \models \varphi[a_1, \ldots, a_n] \quad \text{implies} \quad \mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n].$$

For example, consider the following simple positive bounded sentence σ, where x, y are variables of sort s:

$$\exists_1 x \exists_1 y (\|x\| = 1 \wedge \|y\| = 1 \wedge \|x + y\| = 1 \wedge \|x - y\| = 1).$$

It is easy to show that $\mathcal{M} \models \sigma$ is true if and only if the normed space $M^{(s)}$ contains a subspace linearly isometric to ℓ^2_∞. On the other hand, $\mathcal{M} \models_{\mathcal{A}} \sigma$ holds if and only if ℓ^2_∞ is finitely representable in $M^{(s)}$ (*i.e.*, for each $\epsilon > 0$, the normed space $M^{(s)}$ has a subspace $(1+\epsilon)$-isomorphic to ℓ^2_∞. Finally, it is easy to construct Banach spaces X such that ℓ^2_∞ is finitely representable in X while X contains no subspace linearly isometric to ℓ^2_∞. If X is such a space, $X \models_{\mathcal{A}} \sigma$ while $X \not\models \sigma$.

Alternative Formulation of Quantifiers

There are two essentially equivalent ways to formulate bounded universal quantifiers. Consider a positive bounded formula σ of the form

$$\forall y(\|y\| \leq r \rightarrow \varphi).$$

As the term "positive" is normally used in formal logic, this is not a positive formula, since the atomic formula $\|y\| \leq r$ is implicitly negated due to its position in front of the implication connective. Some logicians may find this to be a bothersome abuse of terminology. However, when a positive bounded formula is interpreted using the semantics of approximate satisfaction, this syntactic issue disappears.

One could systematically replace conditions of the form $\|y\| \leq r$ in the bounded universal quantifier by $\|y\| < r$, thus making the formulas considered in this theory "positive" in the usual technical sense; let us write $\widetilde{\psi}$ for the result of making this change in every bounded universal quantifier in a positive bounded formula ψ. Thus we can write $\widetilde{\sigma}$ as

$$\forall y(\|y\| < r \rightarrow \widetilde{\varphi}).$$

An easy argument by induction shows that if τ is any approximation of σ, then σ implies $\widetilde{\sigma}$ which in turn implies τ, when they are interpreted in any normed space structure. Therefore we also have τ implies $\widetilde{\tau}$.

Our entire theory could be based on the positive formulas $\widetilde{\sigma}$; we would regard the positive formulas $\widetilde{\tau}$ as the approximations of $\widetilde{\sigma}$. The relationships in the previous paragraph yield that σ is approximately satisfied in a given normed space structure $\mathcal{M}$ if and only if $\widetilde{\sigma}$ is approximately satisfied in $\mathcal{M}$ (in the sense that every approximation $\widetilde{\tau}$ is true in $\mathcal{M}$). Hence there is no difference between σ and $\widetilde{\sigma}$ when the approximate semantics is used.

We chose to formulate this theory so that the two bounded quantifiers $\exists_r y$ and $\forall_r y$ are relativized to the same condition, namely $\|y\| \leq r$, in the belief that this would seem more natural to a wider mathematical audience.

Earlier Approaches

In a slightly different form, positive bounded formulas and approximate satisfaction were introduced for Banach spaces (for the most part considered without additional structure) in [Hen76]; see also [HM83b] and [HH86]. Here we consider a somewhat larger collection of positive bounded formulas than was used in those papers, but the differences are slight and inessential. (See the discussion of alternative formulations in Chapter 11.)

Formal languages for structures included among the ones considered here were also introduced by Krivine [Kri72] [Kri74] and by Stern [Ste74] [Ste75] [Ste76] [Ste77] [Ste78]. The formulas considered in these papers had limited quantifier complexity, either universal, existential, or universal-existential, and the focus of the work was on subspace properties or on relationships such as local reflexivity. These papers did not present a fully developed model theory for Banach spaces nor were they adequate to completely characterize the properties shared by a Banach space and its ultrapowers. These papers do contain many interesting results giving axiomatizations of familiar classes of spaces, and these results can be adapted to yield axiomatizations by positive bounded sentences for the most part. See the remarks in Chapter 13.

6

Basic Model Theory I

6.1 Definition. Let L be a signature for normed space structures. A *positive bounded theory* in L is a set of positive bounded L-sentences (*i.e.*, formulas without free variables). If Γ is a positive bounded theory in L and $\mathcal{M}$ is a normed space L-structure, we say that $\mathcal{M}$ *is an approximate model* of Γ if $\mathcal{M} \models_{\mathcal{A}} \Gamma$. This is equivalent to saying that every sentence in Γ^+ is true in $\mathcal{M}$.

If $\mathcal{M}$ is a normed space L-structure, the *positive bounded theory of* $\mathcal{M}$, denoted $\mathrm{Th}_{\mathcal{A}}(\mathcal{M})$, is the set of positive bounded L-sentences which are approximately satisfied by $\mathcal{M}$.

6.2 Definition. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are normed space L-structures based on $(M^{(s)} \mid s \in S)$ and $(N^{(s)} \mid s \in S)$, respectively.

(i) We say that $\mathcal{M}$ and $\mathcal{N}$ are *approximately elementarily equivalent*, and write

$$\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N},$$

if $\mathcal{M}$ and $\mathcal{N}$ approximately satisfy exactly the same positive bounded L-sentences *i.e.* if $\mathrm{Th}_{\mathcal{A}}(\mathcal{M}) = \mathrm{Th}_{\mathcal{A}}(\mathcal{N})$.

(ii) If $\mathcal{M} \subseteq \mathcal{N}$ we say that $\mathcal{M}$ is an *approximate elementary substructure of* $\mathcal{N}$, and write

$$\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N},$$

if the following condition holds: whenever $\varphi(x_1, \ldots, x_n)$ is a positive bounded L-formula and $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$, we have

$$\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathcal{N} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n].$$

In this case, we also say that $\mathcal{N}$ is an *approximate elementary extension* of $\mathcal{N}$.

(iii) An *approximate elementary embedding* of $\mathcal{M}$ into $\mathcal{N}$ consists of a family of maps $(T^{(s)} \mid s \in S)$ with $T^{(s)}\colon M^{(s)} \to N^{(s)}$ for $s \in S$ such that the following condition holds: whenever $\varphi(x_1, \ldots, x_n)$ is a positive bounded L-formula and $a_1, \ldots, a_n$ are elements of $\mathcal{M}$, where x_k is a variable of sort s_k and $a_k \in M^{(s_k)}$ for $k = 1, \ldots, n$, we have

$$\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad \mathcal{N} \models_{\mathcal{A}} \varphi[T^{(s_1)}(a_1), \ldots, T^{(s_n)}(a_n)].$$

6.3 Remarks.

(i) Every approximate elementary embedding of one normed space structure into another is an embedding.

(ii) The collection of approximate elementary embeddings is closed under composition.

(iii) Every isomorphism between normed space structures is an approximate elementary embedding.

6.4 Proposition. *Let $\mathcal{M}$ and $\mathcal{N}$ be normed space L-structures. The following conditions are equivalent:*

(i) $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$;

(ii) *For every positive bounded L-sentence φ and every approximation φ' of φ,*

$$\mathcal{M} \models \varphi \quad \textit{implies} \quad \mathcal{N} \models \varphi'.$$

Proof. The implication (i)$\Rightarrow$(ii) is trivial. For the converse, note that condition (ii) implies immediately that $\mathrm{Th}_{\mathcal{A}}(\mathcal{M}) \subseteq \mathrm{Th}_{\mathcal{A}}(\mathcal{N})$. To complete the proof suppose that (ii) is true but that there exists a positive bounded sentence ψ such that $\mathcal{N} \models_{\mathcal{A}} \psi$ and $\mathcal{M} \not\models_{\mathcal{A}} \psi$. Then there exists an approximation ψ' of ψ such that $\mathcal{M} \models \operatorname{neg}(\psi')$ by Proposition 5.14. Let σ, σ' be approximations of ψ such that $\psi < \sigma < \sigma' < \psi'$. By 5.13(iii) we have $\operatorname{neg}(\psi') < \operatorname{neg}(\sigma')$. Hence by the assumed condition (ii) we get $\mathcal{N} \models \operatorname{neg}(\sigma')$. By Remark 5.13(iv) we have $\mathcal{N} \not\models \sigma$ which contradicts the assumption that $\mathcal{N} \models_{\mathcal{A}} \psi$. $\square$

6.5 Remark. Proposition 6.4 shows the usefulness of the weak negation operation. This result yields that if $\mathcal{M}$ and $\mathcal{N}$ are normed space L-structures with $\mathrm{Th}_{\mathcal{A}}(\mathcal{M}) \subseteq \mathrm{Th}_{\mathcal{A}}(\mathcal{N})$, then $\mathrm{Th}_{\mathcal{A}}(\mathcal{M}) = \mathrm{Th}_{\mathcal{A}}(\mathcal{N})$ (that is, $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$.

6.6 Proposition (Tarski-Vaught Test for $\preceq_{\mathcal{A}}$). *Suppose $\mathcal{M}$, $\mathcal{N}$ are normed space L-structures with $\mathcal{M} \subseteq \mathcal{N}$. The following conditions are equivalent:*

(i) $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$;

(ii) *For every positive bounded L-formula $\varphi(x_1, \ldots, x_n, y)$ and every approximation φ' of φ the following condition holds: if $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$ and b is in a suitable sort of $\mathcal{N}$, and if*

$$\mathcal{N} \models \varphi[a_1, \ldots, a_n, b],$$

then there exists c in a suitable sort of $\mathcal{M}$ such that

$$\mathcal{N} \models \varphi'[a_1, \ldots, a_n, c].$$

Proof. The implication from the first condition to the second is immediate from the definition of $\preceq_{\mathcal{A}}$. For the converse direction, one proves the equivalence

$$\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathcal{N} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n]$$

(for $a_1, \ldots, a_n$ in suitable sorts of $\mathcal{M}$) by induction on the complexity of φ, using the given condition to cover the case when φ begins with a quantifier. $\square$

6.7 Corollary. *Let $\mathcal{M}$ and $\mathcal{N}$ be normed space L-structures with $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$. Let $\overline{\mathcal{M}}$ be the completion of $\mathcal{M}$ and let $\mathcal{M}'$ be the closure of $\mathcal{M}$ in $\mathcal{N}$. Then*
(i) $\mathcal{M} \preceq_{\mathcal{A}} \overline{\mathcal{M}}$ *and*
(ii) $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{M}' \preceq_{\mathcal{A}} \mathcal{N}$.

Proof. Use the Perturbation Lemma (Proposition 5.15) to verify the second condition in Proposition 6.6. $\square$

7

Quantifier-Free Formulas

In this chapter we consider normed spaces (over $\mathbb{R}$) without any additional structure, and we give some information about what can be expressed by quantifier-free positive bounded formulas.

Here L_0 is the signature for normed spaces. This has two sorts, one of which is for the scalar field $\mathbb{R}$, and function symbols corresponding to the fact that each sort is regarded as a normed space. If X is a normed space, we regard it as an L_0-structure in an obvious way, as follows: the distinguished functions in this L_0-structure include the field operations $+, -, \times$, the identities 0 and 1, and the absolute value function, all on $\mathbb{R}$; in addition there are the vector addition and subtraction functions (mapping $X \times X$ to X), the norm (mapping X to $\mathbb{R}$), the scalar multiplication (mapping $\mathbb{R} \times X$ to X), and the additive identity 0 in X. The signature also has constant symbols naming the rational numbers within the sort $\mathbb{R}$.

Let X and Y be normed spaces, and consider $\bar{a} = (a_1, \dots, a_n) \in X^n$ and $\bar{b} = (b_1, \dots, b_n) \in Y^n$. Let $\lambda \geq 1$. We say $\bar{a}$ and $\bar{b}$ are λ-*equivalent*, and write $\bar{a} \sim_\lambda \bar{b}$, if for every sequence of scalars $r_1, \dots, r_n$ we have

$$\lambda^{-1} \Big\| \sum_{j=1}^{n} r_j a_j \Big\|_X \leq \Big\| \sum_{j=1}^{n} r_j b_j \Big\|_Y \leq \lambda \Big\| \sum_{j=1}^{n} r_j a_j \Big\|_X .$$

Equivalently, $\bar{a} \sim_\lambda \bar{b}$ holds iff there is a linear isomorphism T from the linear span of $\bar{a}$ onto the linear span of $\bar{b}$ satisfying $T(a_j) = b_j$ for all $j = 1, \dots, n$ and with $\|T\| \leq \lambda$ and $\|T^{-1}\| \leq \lambda$. (We refer to a linear isomorphism satisfying these norm bounds as a λ-*isomorphism*.)

7.1 Theorem. *Let $\bar{a} = (a_1, \dots, a_n)$ be a linearly independent sequence in a normed space X and let $1 \leq \lambda < \eta$. There exists a quantifier-free positive bounded L_0-formula φ and an approximation $\varphi' > \varphi$ such that* *for any normed space Y and any $\bar{b} = (b_1, \ldots, b_n) \in Y^n$ the following two conditions hold:*

(i) *If $\bar{a} \sim_\lambda \bar{b}$ then $Y \models \varphi[\bar{b}]$;*
(ii) *if $Y \models \varphi'[\bar{b}]$ then $\bar{b} \sim_\eta \bar{a}$.*

Proof. The construction of φ and φ' is based on a familiar set of estimates and depends on the compactness of closed bounded sets in finite dimensional normed spaces.

Fix $\bar{a} = (a_1, \ldots, a_n)$ linearly independent in a normed space X. Let E be the linear span of $\bar{a}$. Fix r and $\epsilon > 0$ such that $\|a_i\| \leq r$ for all $i = 1, \ldots, n$ and

$$\Big\| \sum_{j=1}^{n} r_j a_j \Big\| \geq \epsilon \cdot \max(|r_1|, \ldots, |r_n|)$$

for all $r_1, \ldots, r_n \in \mathbb{R}$. Fix δ close to 0 in the interval $0 < \delta < 1/2$, with its size to be determined later. We suppose that r, ϵ, δ are all in $\mathbb{Q}$.

Let F be a finite subset of $\mathbb{Q}^n$ such that for each $(r_1, \ldots, r_n) \in F$ we have

$$1 - \delta \leq \Big\| \sum_{j=1}^{n} r_j a_j \Big\| \leq 1 + \delta.$$

We take F large enough so that for any $z \in E$ with $\|z\| = 1$ there exists $(r_1, \ldots, r_n) \in F$ such that

$$\Big\| z - \sum_{j=1}^{n} r_j a_j \Big\| \leq \delta.$$

We let $\varphi(v_1, \ldots, v_n)$ be the conjunction of the atomic formulas $\|v_j\| \leq \lambda r$ for $j = 1, \ldots, n$ as well as the atomic formulas

$$(1 - \delta)\lambda^{-1} \leq \Big\| \sum_{j=1}^{n} r_j v_j \Big\| \leq (1 + \delta)\lambda$$

as $(r_1, \ldots, r_n)$ ranges over F. Note that since L_0 has a constant naming each rational number, this is a well-formed positive bounded quantifier-free formula in L_0.

We let φ' be formed in the same way as φ, except we replace each occurrence of δ by 2δ and we replace $r\lambda$ by $r\lambda + \delta$. It is clear that φ' is an approximation of φ.

The construction makes it clear that φ satisfies condition (i) in the Theorem. Suppose Y is any normed space and $\bar{b} = (b_1, \ldots, b_n)$ is any sequence in Y such that $Y \models \varphi'[b_1, \ldots, b_n]$. We must show that $\bar{b} \sim_\eta \bar{a}$. To do this it suffices to consider any element $z = \sum_{j=1}^n s_j a_j$ with $\|z\| = 1$ and to show that

$$\eta^{-1} \leq \Big\| \sum_{j=1}^n s_j b_j \Big\| \leq \eta.$$

Given such an element z, choose $(r_1, \ldots, r_n) \in F$ such that

$$\Big\| z - \sum_{j=1}^n r_j a_j \Big\| \leq \delta.$$

This sequence determines one of the atomic formulas that we included in φ', from which we conclude that

$$(1-2\delta)\lambda^{-1} \leq \Big\| \sum_{j=1}^n r_j b_j \Big\| \leq (1+2\delta)\lambda.$$

Our choice of ϵ ensures that

$$\delta \geq \Big\| \sum_{j=1}^n r_j a_j - \sum_{j=1}^n s_j a_j \Big\| \geq \epsilon \cdot \max(|r_j - s_j| \mid j = 1, \ldots, n).$$

Hence, using the triangle inequality we get

$$\Big\| \sum_{j=1}^n r_j b_j - \sum_{j=1}^n s_j b_j \Big\| \leq \frac{\delta}{\epsilon} n(r\lambda + \delta).$$

Let $\gamma = \frac{\delta}{\epsilon} n(r\lambda + \delta)$. Note that γ can be made arbitrarily small by choosing δ small enough. From these estimates we conclude that

$$(1-2\delta)\lambda^{-1} - \gamma \leq \Big\| \sum_{j=1}^n s_j b_j \Big\| \leq (1+2\delta)\lambda + \gamma.$$

This makes it clear that φ' satisfies (ii) as long as we choose δ small enough so that $(1+2\delta)\lambda + \gamma \leq \eta$ and $(1-2\delta)\lambda^{-1} - \gamma \geq \eta^{-1}$.

□

Note that the preceding Theorem cannot be true in precisely this form if $\bar{a} = (a_1, \ldots, a_n)$ is linearly dependent. Suppose otherwise, that formulas $\varphi' > \varphi$ exist satisfying the conditions of the Theorem. We may consider $\bar{a}$ embedded isometrically in an infinite dimensional normed space X. Let $\delta > 0$ be as in the Perturbation Lemma (Proposition 5.15) for the formulas $\varphi' > \varphi$ and for $r = \max(\|a_1\|, \ldots, \|a_n\|)$. There exists a perturbation $\bar{b}$ of $\bar{a}$ in X such that $\bar{b}$ is linearly independent and such that $\|b_j - a_j\| < \delta$ for all $j = 1, \ldots, n$. Since $X \models \varphi[\bar{a}]$ it follows from the choice of δ that $X \models \varphi'[\bar{b}]$. Therefore $\bar{a}$ is isomorphically equivalent to $\bar{b}$, from which it follows that $\bar{a}$ must be linearly independent.

However, the quantifier-free formulas shown to exist in the preceding Theorem are sufficient for most purposes. We will next use them to show that whenever $X \preceq_{\mathcal{A}} Y$, the normed spaces X and Y have the same "local reflexivity" relationship that X has with its second dual. (However, there is a limit to this analogy; indeed, the second dual of X is rarely an approximate elementary extension of X.)

7.2 Corollary. *Let X and Y be normed spaces with $X \preceq_{\mathcal{A}} Y$. Let E be a finite dimensional subspace of Y and $\eta > 1$. There exists an η-isomorphism T of E into X such that $T(x) = x$ for all $x \in E \cap X$. In particular, Y is finitely representable in X.*

Proof. Let X, Y, E, λ be as stated. Choose a basis $a_1, \ldots, a_m$ for E such that $a_1, \ldots, a_k$ is a basis for $E \cap X$, $k \leq m$. Taking $\lambda = 1$, let $\varphi < \varphi'$ be quantifier-free positive bounded formulas as in Theorem 7.1. Let $r = \max(\|a_{k+1}\|, \ldots, \|a_m\|)$ and $s > r$. By the choice of φ we have $Y \models \exists_r x_{k+1} \ldots \exists_r x_m \varphi[a_1, \ldots, a_k]$. Since $X \preceq_{\mathcal{A}} Y$ this implies $X \models \exists_s x_{k+1} \ldots \exists_s x_m \varphi'[a_1, \ldots, a_k]$. Hence there exists $b_{k+1}, \ldots, b_m \in X$ such that $X \models \varphi'[a_1, \ldots, a_k, b_{k+1}, \ldots, b_m]$. Therefore

$$(a_1, \ldots, a_k, a_{k+1}, \ldots, a_m) \sim_\eta (a_1, \ldots, a_k, b_{k+1}, \ldots, b_m)$$

by the choice of φ'. That is, there is an η-isomorphism T such that for $j = 1, \ldots, m$ we have $T(a_j) = a_j$ and for $j = k+1, \ldots, m$ we have $T(a_j) = b_j$. Therefore T is the desired η-isomorphism of E into X. $\square$

In the case where Y is an ultrapower or a nonstandard hull of X, the preceding result was proved independently by Stern [Ste76] and by Henson and Moore [HM74a] [HM74b].

Note the following consequence of Corollary 7.2: suppose $\mathcal{M}$ and $\mathcal{N}$ are normed space L-structures and $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$. If $s \in S$ and $M^{(s)}$ is finite dimensional, then $N^{(s)} = M^{(s)}$.

8

Ultraproducts of Normed Space Structures

Uniform Families of L-structures

8.1 Definition. Suppose that $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a family of normed space L-structures and that $\mathcal{M}_\xi$ is based on $(M_\xi^{(s)} \mid s \in S)$ for each $\xi \in \Lambda$. Let $f\colon s_1 \times \cdots \times s_m \to s_0$ be a function symbol of L, so $f^{\mathcal{M}_\xi}$ denotes the interpretation of f in $\mathcal{M}_\xi$. We say that $(f^{\mathcal{M}_\xi} \mid \xi \in \Lambda)$ is a *uniform family of functions* if the following two conditions hold. (They express the requirement that the functions $f^{\mathcal{M}_\xi}$ are bounded and uniformly continuous on bounded subsets of their domains in a way that is *uniform* in the index $\xi \in \Lambda$.)

(i) (Uniform Boundedness Condition) For each sequence of positive real numbers $r_1, \ldots, r_m$ there exists a positive real number R such that for arbitrary $\xi \in \Lambda$ and

$$x_\xi^k \in \mathcal{B}_{r_k}(M_\xi{}^{(s_k)}), \qquad k = 1, \ldots, m,$$

we have

$$\|f^{\mathcal{M}_\xi}(x_\xi^1, \ldots, x_\xi^m)\| \leq R.$$

When this holds, the number R is called a *uniform bound for* $(f_\xi^{\mathcal{M}} \mid \xi \in \Lambda)$ *on* $(\ \mathcal{B}_{r_1}(M_\xi{}^{(s_1)}) \times \cdots \times \mathcal{B}_{r_m}(M_\xi{}^{(s_m)}) \mid \xi \in \Lambda\ )$.

(ii) (Uniform Continuity Condition) For each sequence of positive real numbers $r_1, \ldots, r_m$ there exists a function $\Delta\colon (0,\infty) \to (0,\infty)$ such that for every $\xi \in \Lambda$, Δ is a modulus of uniform continuity for $f^{\mathcal{M}_\xi}$ on $\mathcal{B}_{r_1}(M_\xi{}^{(s_1)}) \times \cdots \times \mathcal{B}_{r_m}(M_\xi{}^{(s_m)})$ (see Definition 4.2). In this case, we say that Δ is *uniform modulus of uniform continuity for* $(f_\xi^{\mathcal{M}} \mid \xi \in \Lambda)$ *on* $(\ \mathcal{B}_{r_1}(M_\xi{}^{(s_1)}) \times \cdots \times \mathcal{B}_{r_m}(M_\xi{}^{(s_m)}) \mid \xi \in \Lambda\ )$.

We say that a family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ of normed space L-structures is *uniform* if for every function symbol f of L, the family of functions $(f^{\mathcal{M}_\xi} \mid \xi \in \Lambda)$ is uniform.

A class $\mathfrak{C}$ of normed space L-structures is called *uniform* if every family of structures in $\mathfrak{C}$ is uniform.

8.2 Remark. Suppose that $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a family of normed space L-structures and that $\mathcal{M}_\xi$ is based on $(M_\xi^{(s)} \mid s \in S)$ for each $\xi \in \Lambda$. Let $f\colon s_1 \times \cdots \times s_m \to s_0$ be a function symbol of L. Suppose the uniform continuity condition holds for $(f_\xi^{\mathcal{M}} \mid \xi \in \Lambda)$ on $(\ \mathcal{B}_{r_1}(M_\xi{}^{(s_1)}) \times \cdots \times \mathcal{B}_{r_m}(M_\xi{}^{(s_m)}) \mid \xi \in \Lambda\ )$. In order for the uniform boundedness condition to hold for $(f_\xi^{\mathcal{M}} \mid \xi \in \Lambda)$ on $(\ \mathcal{B}_{r_1}(M_\xi{}^{(s_1)}) \times \cdots \times \mathcal{B}_{r_m}(M_\xi{}^{(s_m)}) \mid \xi \in \Lambda\ )$ it suffices for the family $(\|f^{\mathcal{M}_\xi}(0,\dots,0)\| \mid \xi \in \Lambda)$ to be bounded in $\mathbb{R}$..

In particular, if f is a constant symbol, $(f^{\mathcal{M}_\xi} \mid \xi \in \Lambda)$ is uniform iff there is a uniform bound on $(\|f^{\mathcal{M}_\xi}\| \mid \xi \in \Lambda)$ (*i.e.*, the uniform continuity requirement is automatic).

8.3 Examples. The following are some examples of uniform classes of L-structures, for various signatures L:

(i) Normed spaces over $\mathbb{R}$.
(ii) Normed spaces over $\mathbb{C}$.
(iii) Normed vector lattices.
(iv) Normed algebras.
(v) C^*-algebras.
(vi) Hilbert spaces (with inner product included as a function).
(vii) Dual pairs (X, X'), where X is a Banach space, X' is the dual of X, and the pairing is included as a function.
(viii) Triples (X, X', X''), where X' and X'' are the dual and the double dual of X and the pairing between X and X', the pairing between X' and X'', and the embedding $X \to X''$ are included as functions.
(ix) Operator spaces (with the infinite family of matrix norms included as functions).
(x) Any uniform class $\mathfrak{C}$ of normed space structures can be expanded to a uniform class by adding to each structure in $\mathfrak{C}$ a list of bounded linear operators as functions, as long as for each additional function symbol f there is a uniform bound on the operator norms of $f^{\mathcal{M}}$ as $\mathcal{M}$ ranges over $\mathfrak{C}$.

8.4 Remarks. Let $\mathfrak{C}$ be a uniform class of normed space L-structures. There exists a set Γ of positive bounded sentences which expresses the specific conditions on $\mathfrak{C}$ that ensure it is uniform. That is, every structure in $\mathfrak{C}$ is an approximate model of Γ and the class of all approximate models of Γ is uniform itself. Moreover, the sentences in Γ can all be taken to be in prenex normal form with only $\forall$ quantifiers. The following two statement follow from this:

(i) If $\mathfrak{D}$ is the class of all L-structures isomorphic to a substructure of some structure in $\mathfrak{C}$, then $\mathfrak{D}$ is also a uniform class of L-structures.

(ii) If $\mathfrak{D}$ is the class of all L-structures that are approximately elementarily equivalent to some member of $\mathfrak{C}$, then $\mathfrak{D}$ is also a uniform class of L-structures.

Ultraproducts of Normed Space L-structures

Suppose that $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a family of normed space L-structures and that $\mathcal{M}_\xi$ is based on $(M_\xi^{(s)} \mid s \in S)$ for each $\xi \in \Lambda$. Let $f\colon s_1 \times \cdots \times s_m \to s_0$ be a function symbol of L such that the family $(f^{\mathcal{M}_\xi} \mid \xi \in \Lambda)$ of interpretations of f is uniform. Given an ultrafilter D on Λ, we define a function

$$\Big(\prod_{\xi\in\Lambda} f^{M_\xi}\Big)_D \colon \Big(\prod_{\xi\in\Lambda} M_\xi^{(s_1)}\Big)_D \times \cdots \times \Big(\prod_{\xi\in\Lambda} M_\xi^{(s_m)}\Big)_D \to \Big(\prod_{\xi\in\Lambda} M_\xi^{(s_0)}\Big)_D$$

as follows. If $(x_\xi^k)_{\xi\in\Lambda} \in \ell_\infty(\Lambda, (M_\xi^{(s_k)} \mid \xi \in \Lambda))$ for $k = 1, \ldots, m$, we let

$$\Big(\prod_{\xi\in\Lambda} f^{M_\xi}\Big)_D (((x_\xi^1)_{\xi\in\Lambda})_D, \ldots, ((x_\xi^m)_{\xi\in\Lambda})_D) = \Big((f^{M_\xi}(x_\xi^1, \ldots, x_\xi^m))_{\xi\in\Lambda}\Big)_D.$$

The fact that the family $(f^{\mathcal{M}_\xi} \mid \xi \in \Lambda)$ is uniform ensures that the function $(\prod_{\xi\in\Lambda} f^{M_\xi})_D$ is well-defined. Note that any uniform modulus of uniform continuity for $(f^{M_\xi} \mid \xi \in \Lambda)$ on

$$(\ \mathcal{B}_{r_1}(M_\xi{}^{(s_1)}) \times \cdots \times \mathcal{B}_{r_m}(M_\xi{}^{(s_m)}) \mid \xi \in \Lambda\ )$$

is also a modulus of uniform continuity for $(\prod_{\xi\in\Lambda} f^{M_\xi})_D$ on

$$\mathcal{B}_{r_1}(\Big(\prod_{\xi\in\Lambda} M_\xi{}^{(s_1)}\Big)_D) \times \cdots \times \mathcal{B}_{r_m}(\Big(\prod_{\xi\in\Lambda} M_\xi{}^{(s_m)}\Big)_D).$$

Similarly, any uniform bound for $(f^{M_\xi} \mid \xi \in \Lambda)$ yields a bound for $(\prod_{\xi\in\Lambda} f^{M_\xi})_D$.

8.5 Definition. If $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a uniform family of normed space L-structures such that $\mathcal{M}_\xi$ is based on $(M_\xi^{(s)} \mid s \in S)$ and D is an ultrafilter on Λ, the D-*ultraproduct* of $(\mathcal{M}_\xi \mid \xi \in \Lambda)$, denoted

$$\Big(\prod_{\xi\in\Lambda} \mathcal{M}_\xi\Big)_D$$

is the L-structure $\mathcal{M}$ defined as follows. The sorts of $\mathcal{M}$ are the normed space D-ultraproducts

$$\mathcal{M}^{(s)} = \Big(\prod_{\xi\in\Lambda} M_\xi^{(s)}\Big)_D$$

for each $s \in S$, and

$$f^{\mathcal{M}} = \Big(\prod_{\xi\in\Lambda} f^{\mathcal{M}_\xi}\Big)_D$$

for each function symbol $f\colon s_1 \times \cdots \times s_m \to s_0$ of L.

8.6 Remarks.

(i) The definition of uniform class of normed space structures is precisely what is needed to ensure that the normed space ultraproduct construction makes sense when applied to any family taken from the class. That is, if $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a family of normed space L-structures, then the definition of $(\prod_{\xi\in\Lambda} \mathcal{M})_D$ makes sense for every ultrafilter D on Λ if and only if $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is uniform.

(ii) Many operations on normed space structures preserve uniformity. For example, if $\mathfrak{C}$ is a uniform class of structures, then the class $\mathfrak{D}$ that results from closing $\mathfrak{C}$ under ultraproducts is also uniform. See also Remark 8.4.

Nonstandard Hulls

Let D be an ultrafilter on the index set Λ and let $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ be a uniform family of normed space L-structures. As indicated on page 19 for ultrapowers, the ultraproduct $(\prod_{\xi\in\Lambda} \mathcal{M})_D$ can be seen as a nonstandard hull, when the superstructure V and nonstandard extension ${}^*V$ are suitably chosen.

Namely, let V be any set theoretic superstructure which contains the set $S = \{\mathcal{M}_\xi \mid \xi \in \Lambda\}$ as an element. Let ${}^*V$ be the D-ultrapower nonstandard extension of V. The family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ corresponds to an internal element M of ${}^*S$; in particular, M is an internal normed space L-structure. For each function symbol f of L, the interpretation f^M of f in M is the internal function whose representative is the family $(f^{\mathcal{M}_\xi} \mid \xi \in \Lambda)$. The requirement that the family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ be uniform ensures that for each function symbol f of L, the internal function f^M takes finite arguments to finite values, and preserves the infinitesimal equivalence relation. Hence M gives rise to a natural quotient structure (keeping finite elements of M and identifying two of them if their difference is infinitesimal) which is a normed space L-structure, denoted by $\widehat{M}$. The nonstandard hull $\widehat{M}$ is isomorphic, as a normed space L-structure, to the ultraproduct $(\prod_{\xi\in\Lambda} \mathcal{M})_D$.

9

Basic Model Theory II

In this chapter we continue the development of basic model theory for normed space structures, including such fundamental results as the Compactness Theorem (Propositions 9.5 and 9.22).

Our first result gives a uniform version of the Perturbation Lemma (Proposition 5.15), strengthening a key property of positive bounded formulas and approximate satisfaction.

9.1 Proposition (Perturbation Lemma for Uniform Classes). *Suppose that $\mathfrak{C}$ is a uniform class of L-structures and $\varphi(x_1, \ldots, x_n)$ is a positive bounded L-formula. Then for every $r > 0$ and every approximation φ' of φ there exists $\delta > 0$ such that whenever $\mathcal{M}$ is a structure in $\mathfrak{C}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are elements of suitable sorts of $\mathcal{M}$, we have that*

$$\mathcal{M} \models \bigwedge_{1 \leq i \leq n} \|a_i\| \leq r \ \wedge \ \bigwedge_{1 \leq i \leq n} \|a_i - b_i\| \leq \delta \ \wedge \ \varphi[a_1, \ldots, a_n]$$

implies

$$\mathcal{M} \models \varphi'[b_1, \ldots, b_n].$$

Proof. The proof of Proposition 5.15 depends only on being given specific moduli of uniform continuity and uniform bounds for each of the functions in the structure being considered. Therefore, if all structures considered come from a uniform class of L-structures, then the dependence of δ on r, φ, and φ' can be taken to be the same for all of them. $\square$

Ultraproducts and $\models_{\mathcal{A}}$

The next result is the analogue, in the model theory of normed space structures, of the result in first-order model theory that is known as the *Fundamental Theorem on Ultraproducts* or as the *Łoś Theorem*, and which was proved by J. Łoś [Łoś55].

9.2 Proposition. *Let* $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ *be a uniform family of normed space* L*-structures and let* $\varphi(x_1, \dots, x_n)$ *be a positive bounded formula, where* x_k *is a variable of sort* s_k*, for* $k = 1, \dots, n$*. If* $\mathcal{M}_\xi$ *is based on* $(M_\xi^{(s)} \mid s \in S)$ *and* $a_k \in \ell_\infty(\Lambda, (M_\xi^{(s_k)} \mid \xi \in \Lambda))$*, for* $k = 1, \dots, n$*, then*

$$\Big(\prod_{\xi\in\Lambda} \mathcal{M}_\xi\Big)_D \models_{\mathcal{A}} \varphi[\,(a_1)_D, \dots, (a_n)_D\,]$$

if and only if for every approximation φ' *of* φ*, the set*

$$\{\, \xi \in \Lambda \mid \mathcal{M}_\xi \models \varphi'[a_1(\xi), \dots, a_n(\xi)]\,\}$$

is in the ultrafilter D*.*

Proof. The proof is by induction on the complexity of φ. Use Proposition 5.14. □

9.3 Corollary. *If* $\mathcal{M}$ *is a normed space structure based on* $(M^{(s)} \mid s \in S)$ *and* $T^{(s)}\colon M^{(s)} \to (M^{(s)})_D$ *is the diagonal embedding for each* $s \in S$*, then the family* $(T^{(s)} \mid s \in S)$ *is an approximate elementary embedding of* $\mathcal{M}$ *into* $(\mathcal{M})_D$*.*

Proof. From Proposition 9.2. □

9.4 Corollary. *If* $\mathcal{M}$ *and* $\mathcal{N}$ *are normed space* L*-structures and they have isomorphic ultrapowers, then* $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$*.*

The converse of the preceding result is also true, in a strong form: in particular, if $\mathcal{M}$ and $\mathcal{N}$ are normed space L-structures and $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$, then there exists an ultrafilter D such that $(\mathcal{M})_D$ is isomorphic to $(\mathcal{N})_D$. Chapter 10 below is devoted to proving a series of results of this kind.

In the rest of this chapter we regularly use notation established in Definition 3.7, which we now recall. Let L be a signature and $L(C)$ an extension of L by constants, with C being the set of new constants that has been added to L. For any L-structure $\mathcal{M}$, an arbitrary expansion of $\mathcal{M}$ to an $L(C)$-structure is denoted $(\mathcal{M}, \mathbf{A})$; $\mathbf{A}(c)$ is taken to be the interpretation of the constant symbol c in this expansion, for each $c \in C$. The function $\mathbf{A}$ is referred to as an *interpretation of* C *in* $\mathcal{M}$. Note that each constant c must be assigned a unique sort from L in order to specify the signature $L(C)$, and the function $\mathbf{A}$ must respect these sort assignments.

The Compactness Theorem

9.5 Proposition (Compactness Theorem). *Let Γ be a set of positive bounded L-sentences. Suppose that $\mathfrak{C}$ is a uniform class of normed space L-structures such that for each finite subset Σ of Γ^+ there exists a structure in $\mathfrak{C}$ which satisfies Σ. Then there exists an ultraproduct of structures from $\mathfrak{C}$ which approximately satisfies the entire set Γ.*

Proof. Let Λ be the set of finite subsets of Γ^+ and for each $\xi \in \Lambda$ let $\mathcal{M}_\xi$ be a normed space L-structure in $\mathfrak{C}$ which satisfies every sentence in ξ. For every finite subset Δ of Γ^+ let F_Δ be the set of all $\xi \in \Lambda$ such that $\Delta \subseteq \xi$. Note that F_Δ is contained in $\{\xi \in \Lambda \mid \mathcal{M}_\xi \models \Delta\}$. The family $\mathcal{F}$ of sets of the form F_Δ is closed under finite intersections. Then

$$\Big(\prod_{\xi\in\Lambda} \mathcal{M}_\xi\Big)_D \models_{\mathcal{A}} \Gamma$$

for any ultrafilter D on Λ extending $\mathcal{F}$, by Proposition 9.2. $\square$

9.6 Remark. We will see in Proposition 9.22 that the ultraproduct constructed in the proof of Proposition 9.5 actually *satisfies* (not just approximately satisfies) every sentence in Γ.

9.7 Corollary. *Let $\mathfrak{C}$ be a uniform class of normed space L-structures which is closed under ultraproducts. The following conditions are equivalent for a positive bounded theory Γ in L:*

(i) *There exists a structure $\mathcal{M}$ in $\mathfrak{C}$ such that $\Gamma = \mathrm{Th}_{\mathcal{A}}(\mathcal{M})$;*

(ii) *Every finite subset of Γ^+ is satisfied by some structure in $\mathfrak{C}$ and for every positive bounded L-sentence φ and every $\varphi' > \varphi$ either $\varphi \in \Gamma$ or $\mathrm{neg}(\varphi') \in \Gamma$.*

Proof. The implication (i)$\Rightarrow$(ii) follows from Proposition 5.14. To prove (ii)$\Rightarrow$(i), use Proposition 9.5 to obtain a structure $\mathcal{M}$ in $\mathfrak{C}$ such that $\mathcal{M} \models_{\mathcal{A}} \Gamma$. If (i) fails to hold, then there is a positive bounded sentence φ such that $\mathcal{M} \models_{\mathcal{A}} \varphi$ but $\varphi \notin \Gamma$. Let φ', φ'' be approximations of φ with $\varphi < \varphi' < \varphi''$. By (ii) we have $\mathrm{neg}(\varphi'') \in \Gamma$ and hence $\mathcal{M} \models_{\mathcal{A}} \mathrm{neg}(\varphi'')$. By 5.13(iv) we conclude $\mathcal{M} \not\models \varphi'$ which contradicts our assumption that $\mathcal{M} \models_{\mathcal{A}} \varphi$. $\square$

9.8 Definition. A set Γ of positive bounded L-sentences that satisfies $\Gamma = \mathrm{Th}_{\mathcal{A}}(\mathcal{M})$ for some L-structure $\mathcal{M}$ is called a *complete* positive bounded theory in L.

9.9 Proposition. *Let $\mathcal{M}$ and $\mathcal{N}$ be normed space L-structures based on $(M^{(s)} \mid s \in S)$ and $(N^{(s)} \mid s \in S)$, respectively, such that $\mathcal{M} \subseteq \mathcal{N}$. The following conditions are equivalent:*

(i) $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$;

(ii) *There exists an ultrapower $(\mathcal{M})_D$ of $\mathcal{M}$ and an elementary embedding $(T^{(s)} \mid s \in S)$ of $\mathcal{N}$ into $(\mathcal{M})_D$ such that $T^{(s)}$ is the identity on $M^{(s)}$ for each $s \in S$.*

Proof. The proof of (ii)⇒(i) is routine. We give the proof of (i)⇒(ii).

To begin, we add to L a set C of new constants of suitable sorts, with the intention of naming the elements of $\mathcal{N}$. Let $\mathbf{A}$ be an interpretation of C in $\mathcal{N}$ such that every element of $\mathcal{N}$ is of the form $\mathbf{A}(c)$ for some $c \in C$. We consider the $L(C)$-structure $(\mathcal{N}, \mathbf{A})$ and let $\Gamma = \mathrm{Th}_{\mathcal{A}}(\mathcal{N}, \mathbf{A})$.

For each $c \in C$ let r_c be a real number satisfying $\|\mathbf{A}(c)\| < r_c$. Let $\mathfrak{C}$ be the class of all $L(C)$-structures of the form $(\mathcal{M}, \mathbf{B})$ such that for all $c \in C$ we have $\|\mathbf{B}(c)\| < r_c$ as well as having $\mathbf{B}(c) = \mathbf{A}(c)$ in every case where $\mathbf{A}(c)$ is an element of $\mathcal{M}$. Note that $\mathfrak{C}$ is a uniform class of $L(C)$-structures.

The hypothesis $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$ ensures that each finite subset of Γ^+ is satisfied by some member of $\mathfrak{C}$. Using the Compactness Theorem (Proposition 9.5) we obtain a family of structures from $\mathfrak{C}$ and an ultrafilter D such that the D-ultraproduct of the family approximately satisfies the entire set Γ. This ultraproduct is of the form $((\mathcal{M})_D, \mathbf{B})$ for some function $\mathbf{B}$ from C to elements of $(\mathcal{M})_D$. Moreover, if $c \in C$ and $a = \mathbf{A}(c)$ is an element of $\mathcal{M}$, then $B(c)$ must be the image of the diagonal embedding (of $\mathcal{M}$ into $(\mathcal{M})_D$) applied to a. The desired elementary embedding of $\mathcal{N}$ into $(\mathcal{M})_D$ is obtained by mapping $\mathbf{A}(c)$ to $\mathbf{B}(c)$ for each $c \in C$. ☐

Unions of Chains

If Λ is a linearly ordered set, a Λ-*chain* of normed space L-structures is a family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ such that $\mathcal{M}_\xi \subseteq \mathcal{M}_\eta$ for $\xi < \eta$. If $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a chain of L-structures such that the family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is uniform, we can define the *union* $\bigcup_{\xi \in \Lambda} \mathcal{M}_\xi$ as an L-structure as follows. If $\mathcal{M}_\xi$ is based on $(M_\xi^{(s)} \mid s \in S)$ for each $\xi \in \Lambda$, then $\bigcup_{\xi \in \Lambda} \mathcal{M}_\xi$ is based on $(\bigcup_{\xi \in \Lambda} M_\xi^{(s)} \mid s \in S)$, and if $f \colon s_1 \times \cdots \times s_m \to s_0$ is a function symbol of L, then the interpretation of f in $\bigcup_{\xi \in \Lambda} \mathcal{M}_\xi$ is the unique function

$$F\colon \bigcup_{\xi\in\Lambda} M_\xi^{(s_1)} \times \cdots \times \bigcup_{\xi\in\Lambda} M_\xi^{(s_m)} \to \bigcup_{\xi\in\Lambda} M_\xi^{(s_0)}$$

such that for $\xi \in \Lambda$ and $(x_1, \ldots, x_m) \in M_\xi^{(s_1)} \times \cdots \times M_\xi^{(s_m)}$ one has $F(x_1, \ldots, x_m) = f^{\mathcal{M}_\xi}(x_1, \ldots, x_m)$.

The assumption that the chain $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a uniform family of L-structures is needed to ensure that the functions defined above are uniformly continuous on bounded subsets of their domains.

9.10 Definition. A chain of structures $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is called an *approximate elementary chain* if $\mathcal{M}_\xi \preceq_{\mathcal{A}} \mathcal{M}_\eta$ for all $\xi < \eta$.

9.11 Remark. If $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is an approximate elementary chain, then the family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is uniform, so the union $\bigcup_{\xi\in\Lambda} \mathcal{M}_\xi$ is well-defined.

9.12 Corollary. *If $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is an approximate elementary chain, then $\mathcal{M}_\xi \preceq_{\mathcal{A}} \bigcup_{\xi\in\Lambda} \mathcal{M}_\xi$ for every $\xi \in \Lambda$.*

Proof. Use Proposition 6.6. □

Löwenheim-Skolem Theorems

Recall that the *density character* of a topological space is the smallest cardinality of a dense subset of the space. For example, a space is separable if and only if its density character is $\aleph_0$. If A is a topological space, we denote its density character by $\mathrm{density}(A)$.

9.13 Proposition (Downward Löwenheim-Skolem Theorem). *Let $\mathcal{M}$ be a normed space L-structure based on $(M^{(s)} \mid s \in S)$. Suppose that we are given sets $A^{(s)} \subseteq M^{(s)}$ for every $s \in S$ and a cardinal number κ such that* $\mathrm{density}(A^{(s)}) \leq \kappa$ *for every $s \in S$ and* $\mathrm{card}(L) \leq \kappa$*. Then there exists a closed substructure $\mathcal{N}$ of $\mathcal{M}$ based on $(N^{(s)} \mid s \in S)$ such that*

(i) $\mathcal{N} \preceq_{\mathcal{A}} \mathcal{M}$*;*
(ii) $A^{(s)} \subseteq N^{(s)} \subseteq M^{(s)}$*, for every $s \in S$;*
(iii) $\mathrm{density}(N^{(s)}) \leq \kappa$*, for every $s \in S$.*

Proof. For each $s \in S$ let $A_0^{(s)}$ be a dense subset of $A^{(s)}$ of cardinality at most κ. Without loss of generality we may assume that $A_0^{(s_\mathbb{R})}$ contains the set of rational numbers. By suitably enlarging these sets, we may obtain a family of sets $(N_0^{(s)} \mid s \in S)$ such that $A_0^{(s)} \subseteq N_0^{(s)} \subseteq M^{(s)}$ and $\mathrm{card}(N_0^{(s)}) \leq \kappa$ for every $s \in S$ and such that the following closure condition also holds: for every positive bounded formula $\varphi(x_1, \ldots, x_n, x_{n+1})$ and every approximation φ' of φ, with x_k a variable of sort s_k for $k = 1, \ldots, n+1$, if $\mathcal{M} \models \varphi[a_1, \ldots, a_n, a_{n+1}]$ with $a_k \in N_0^{(s_k)}$ for $k = 1, \ldots, n$ and $a_{n+1} \in M^{(s_{n+1})}$, then there exists $b \in N_0^{(s_{n+1})}$ such that $\mathcal{M} \models \varphi'[a_1, \ldots, a_n, b]$. It is possible to do this while maintaining the claimed cardinality bounds because L has at most κ many positive bounded formulas.

For each $s \in S$ we now let $N^{(s)}$ be the closure of $N_0^{(s)}$ in $M^{(s)}$. By considering atomic formulas in the closure condition above, one shows that there is a closed substructure $\mathcal{N}$ of $\mathcal{M}$ that is based on $(N^{(s)} \mid s \in S)$. Using the Perturbation Lemma (Proposition 5.15) one uses the closure condition above to show that $\mathcal{N} \subseteq \mathcal{M}$ satisfy the Tarski-Vaught test (Proposition 6.6) and hence that $\mathcal{N} \preceq_{\mathcal{A}} \mathcal{M}$. □

9.14 Proposition (Upward Löwenheim-Skolem Theorem). *Let $\mathcal{M}$ be a normed space L-structure based on $(M^{(s)} \mid s \in S)$. Suppose that κ is a cardinal number such that $\kappa \geq \mathrm{card}(L)$ and $\kappa \geq \mathrm{density}(M^{(s)})$ for every $s \in S$. Then there exists an approximate elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that every infinite dimensional sort of $\mathcal{N}$ has density character equal to κ; $\mathcal{N}$ may be taken to be a Banach space structure.*

Proof. Let S_1 be the set of $s \in S$ such that $M^{(s)}$ is infinite dimensional. For each $s \in S$, let $M_0^{(s)}$ be a dense subset of $M^{(s)}$ of cardinality at most κ. Add to L a set C of new constants of suitable sorts, with the intention of naming the elements of the sets $M_0^{(s)}$; we require $\mathrm{card}(C) \leq \kappa$. Let $\mathbf{A}$ be an interpretation of C in $\mathcal{M}$ such that for each $s \in S$, every element of $M_0^{(s)}$ is of the form $\mathbf{A}(c)$ for some $c \in C$ of sort s. Extend the signature further by adding for each $s \in S_1$ a set $\{ c_i^s \mid i < \kappa \}$ of new constant symbols of sort s. Let $L(C')$ be the resulting extension by constants of L.

We consider the following positive bounded theory in $L(C')$:

$$\begin{aligned}\Gamma = {} & \mathrm{Th}_{\mathcal{A}}(\mathcal{M}, \mathbf{A}) \\ & \cup \{ \|c_i^s\| = 1 \mid i < \kappa,\ s \in S_1 \} \\ & \cup \{ \|c_i^s - c_j^s\| = 1 \mid i < j < \kappa,\ s \in S_1 \}.\end{aligned}$$

Every finite subset of Γ^+ is satisfied by some expansion of $(\mathcal{M}, \mathbf{A})$ to $L(C')$. Moreover we can require that the interpretations of the constant symbols c_i^s are all of norm at most 1, so the set of expansions needed can be taken to be uniform. The Compactness Theorem (Proposition 9.5) yields an $L(C')$-structure $\mathcal{N}$ such that $\mathcal{N} \models_{\mathcal{A}} \Gamma$; Replacing $\mathcal{N}$ by its completion and using Corollary 6.7 we may take $\mathcal{N}$ to be a Banach space structure. By renaming the elements of $\mathcal{N}$ if necessary, we may assume that $c^{\mathcal{N}} = \mathbf{A}(c)$ for all $c \in C$ and that $\mathcal{M}$ is in fact a substructure of $\mathcal{N}$. Since $\mathrm{Th}_{\mathcal{A}}(\,\mathcal{M}, \mathbf{A}\,) \subseteq \Gamma$ it follows that $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$.

It remains to consider the density character of the infinite dimensional sorts of $\mathcal{N}$, which may well be larger than κ. Suppose $\mathcal{N}$ is based on $(N^{(s)} \mid s \in S)$. Clearly, density$(N^{(s)}) \geq \kappa$, for every $s \in S_1$. Applying Proposition 9.13 we may reduce the density character of the infinite dimensional sorts of $\mathcal{N}$ as needed. $\square$

9.15 Remark. Suppose $\mathcal{M}$ and $\mathcal{N}$ are based on $(M^{(s)} \mid s \in S)$ and $(N^{(s)} \mid s \in S)$, respectively, and $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$. Then, for every nonnegative integer n, $M^{(s)}$ is n-dimensional if and only if $N^{(s)}$ is n-dimensional. If $\mathcal{M} \subseteq \mathcal{N}$ or vice versa, this forces $M^{(s)}$ and $N^{(s)}$ to be equal. (See Corollary 7.2.) Therefore, in passing from $\mathcal{M}$ to $\mathcal{N}$ in the Löwenheim-Skolem theorems, the finite dimensional sorts necessarily remain unchanged.

Saturated Structures

9.16 Definition. Let $\Gamma(x_1, \ldots, x_n)$ be a set of L-formulas and let $\mathcal{M}$ be a normed space L-structure. We say that $\Gamma(x_1, \ldots, x_n)$ is *satisfiable in* $\mathcal{M}$ if there exist elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}$ such that $\mathcal{M} \models \Gamma[a_1, \ldots, a_n]$. We say that $\Gamma(x_1, \ldots, x_n)$ is *approximately satisfiable in* $\mathcal{M}$ if there exist elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}$ such that $\mathcal{M} \models_{\mathcal{A}} \Gamma[a_1, \ldots, a_n]$.

Note that Γ is approximately satisfiable in $\mathcal{M}$ if and only if Γ^+ is satisfiable in $\mathcal{M}$.

9.17 Definition. Let $\mathcal{M}$ be a normed space L-structure and let κ be an infinite cardinal. We say that $\mathcal{M}$ is *κ-saturated* if the following condition holds: whenever C is a set of constants such that card$(C) < \kappa$, $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$, and $\Gamma(x_1, \ldots, x_n)$ is a set of positive bounded $L(C)$-formulas containing the formulas $\|x_k\| \leq r_k$ for $k = 1, \ldots, n$ (where $r_1, \ldots, r_n$ are nonnegative rational numbers), if every finite subset of Γ^+ is satisfiable in $(\mathcal{M}, \mathbf{A})$, then the entire set Γ is approximately satisfiable in $(\mathcal{M}, \mathbf{A})$.

It is easy to see that without the norm bound formulas it would not generally be possible for saturated structures to exist. (Consider the set of formulas $\Gamma(x) = \{n \leq \|x\| \mid n \in \mathbb{N}\}$; every finite subset of Γ is satisfiable in any nontrivial normed space, but Γ itself is certainly not approximately satisfiable in any normed space.)

Given any normed space L-structure $\mathcal{M}$ and any infinite cardinal number κ, there exists a κ-saturated $\mathcal{N}$ such that $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$. We prove this fact at the end of the chapter by a unions-of-chains argument. Also, in Chapter 10 we prove that $\mathcal{M}$ has a κ-saturated ultrapower; the proof is a byproduct of proving a difficult theorem about isomorphic ultraproducts.

It is easier to prove the existence of κ-saturated normed space L-structures in the setting where L is countable and $\kappa = \aleph_1$, and we will do that next. Moreover, for many purposes this is the only degree of saturation that one needs for applications.

Recall that an ultrafilter D is said to be *countably incomplete* if D is not closed under countable intersections. It is equivalent to require the existence of elements J_n of D for each $n \in \mathbb{N}$ such that the intersection $\bigcap_{n\in\mathbb{N}} J_n$ is the empty set. Evidently any non-principal ultrafilter on a countable set is countably incomplete.

9.18 Proposition. *Let L be a countable signature and let D be a countably incomplete ultrafilter on a set Λ. Then for every uniform family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$, the ultraproduct $(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D$ is $\aleph_1$-saturated.*

Proof. In order to simplify the notation, we verify that $(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D$ satisfies the condition of Definition 9.17 for $n = 1$ and $r_1 = 1$.

We have to prove the following statement: if C is a countable set of constant symbols, $\mathbf{A}$ is an interpretation of C in $(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D$ and $\Gamma(x)$ is a set of positive bounded $L(C)$-formulas such that every finite subset of

$$\{\, \|x\| \leq 1 \,\} \cup \Gamma(x)$$

is approximately satisfiable in $(\,(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D,\ \mathbf{A}\,)$, then the entire set $\{\, \|x\| \leq 1 \,\} \cup \Gamma(x)$ is approximately satisfied in $(\,(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D,\ \mathbf{A}\,)$. For each $\xi \in \Lambda$ let $\mathbf{A}_\xi$ be an interpretation of C in $\mathcal{M}_\xi$ such that

$$\mathbf{A}(c) = (\,(\mathbf{A}_\xi(c))_{\xi\in\Lambda}\,)_D, \quad \text{for every } c \in C.$$

Note that

$$\Big(\,(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D,\ \mathbf{A}\,\Big) = \Big(\,\prod_{\xi\in\Lambda} (\mathcal{M}_\xi, \mathbf{A}_\xi)\,\Big)_D.$$

Thus, since L is an arbitrary countable signature and C is also countable, it suffices to prove the following simpler statement:

If $\Gamma(x)$ is a set of positive bounded L-formulas and every finite subset of $\{ \|x\| \le 1 \} \cup \Gamma(x)$ is approximately satisfiable in $(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D$, then $\{ \|x\| \le 1 \} \cup \Gamma(x)$ is approximately satisfiable in $(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D$.

Suppose, then, that every finite subset of $\{ \|x\| \le 1 \} \cup \Gamma(x)$ is approximately satisfiable in $(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D$. Since L is countable, we may write

$$\Gamma(x) = \{ \varphi_n(x) \mid n \in \mathbb{N} \}.$$

Now for every $n \in \mathbb{N}$ let $\{ \varphi_n^k \mid n \in \mathbb{N} \}$ be a set of approximations of φ_n such that

- $\varphi_n < \varphi_n^l < \varphi_n^k$, if $k < l$;
- For every approximation ψ of φ_n there exists $k \in \mathbb{N}$ such that $\varphi_n < \varphi_n^k < \psi$.

Since D is countably incomplete, we can fix a descending chain of elements of D

$$\Lambda = \Lambda_0 \supseteq \Lambda_1 \supseteq \dots$$

such that $\bigcap_{k\in\mathbb{N}} \Lambda_k = \emptyset$.

Let $(r_k \mid k \ge 1)$ be a decreasing sequence of real numbers whose limit is 1. Let $X_0 = \Lambda$ and for each positive integer k define

$$X_k = \Lambda_k \cap \ \{ \ \xi \in \Lambda \ \mid \ \mathcal{M}_\xi \models \exists_{r_k} x \, (\, \varphi_1^k(x) \wedge \dots \ \wedge \varphi_k^k(x) \,) \ \}.$$

Then $X_k \in D$ by Proposition 9.2. We have $X_k \supseteq X_{k+1}$ for every $k \in \mathbb{N}$ and $\bigcap_{k\in\mathbb{N}} X_k = \emptyset$, so for each $\xi \in \Lambda$ there exists a largest positive integer $k(\xi)$ such that $\xi \in X_{k(\xi)}$.

Suppose that x is a variable of sort s. We now define an element $a \in \ell_\infty(\Lambda, (M_\xi^{(s)} \mid \xi \in \Lambda))$ such that

$$\Big(\prod_{\xi\in\Lambda} \mathcal{M}_\xi\Big)_D \models_{\mathcal{A}} \|(a)_D\| \le 1, \qquad \Big(\prod_{\xi\in\Lambda} \mathcal{M}_\xi\Big)_D \models_{\mathcal{A}} \Gamma[(a)_D].$$

For $\xi \in \Lambda$, if $k(\xi) = 0$, let $a(\xi) = 0_{M_\xi^{(s)}}$; otherwise, let $a(\xi)$ be such that

$$\mathcal{M}_\xi \models \ \|a(\xi)\| \le 1 \ \wedge \ (\varphi_1^{k(\xi)} \wedge \dots \wedge \varphi_{k(\xi)}^{k(\xi)})[a(\xi)].$$

If $k \in \mathbb{N}$ and $\xi \in X_k$, we have $k \le k(\xi)$, so $\mathcal{M}_\xi \models \varphi_k^{k(\xi)}[a(\xi)]$. It follows from Proposition 9.2 that $(\prod_{\xi\in\Lambda} \mathcal{M}_\xi)_D \models_{\mathcal{A}} \Gamma[(a)_D]$. □

9.19 Remark. Note that the ultrafilters constructed in the proofs of Propositions 9.5 and 9.9 are both countably incomplete. Hence, by

Proposition 9.18, the ultrapowers constructed in those proofs are both $\aleph_1$-saturated.

Now we give some applications of the existence of saturated normed space structures.

9.20 Proposition. *Let $\mathcal{M}$ be an L-structure and $\varphi(x_1, \ldots, x_n)$ a positive bounded L-formula. If $\mathcal{M}$ is $\aleph_1$-saturated, then for any elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}$, we have*

$$\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \qquad \textit{if and only if} \quad \mathcal{M} \models \varphi[a_1, \ldots, a_n].$$

Proof. By induction on the complexity of φ. The $\aleph_1$-saturation is used to handle the case when φ begins with a bounded quantifier. □

9.21 Proposition. *Every $\aleph_1$-saturated normed space structure is complete.*

Proof. Suppose that $\mathcal{M}$ is an $\aleph_1$-saturated L-structure and that (a_n) is a Cauchy sequence in a sort $M^{(s)}$ of $\mathcal{M}$. Without loss of generality, assume $\|a_n\| \leq 1$ for every index n. For each positive ϵ choose a positive integer $N = N(\epsilon)$ such that

$$m, n \geq N \quad \text{implies} \quad \|a_m - a_n\| \leq \epsilon.$$

Let $C = \{\, c_n \mid n \in \mathbb{N} \}$ be a set of new constant symbols of sort s and define a set $\Gamma(x)$ of $L(C)$-formulas as follows.

$$\Gamma(x) = \{\, \|x\| \leq 1 \,\} \cup \{\, \|x - c_n\| \leq \epsilon \mid n \geq N(\epsilon),\, \epsilon > 0 \,\}.$$

Define an interpretation $\mathbf{A}$ of C in $\mathcal{M}$ by letting $\mathbf{A}(c_n) = a_n$. Every finite subset of Γ is approximately satisfiable in $(\mathcal{M}, \mathbf{A})$, so by the $\aleph_1$-saturation of $\mathcal{M}$, the set Γ is approximately satisfiable in $(\mathcal{M}, \mathbf{A})$. If a is the element of $M^{(s)}$ that approximately satisfies $\Gamma(x)$, it is clear that (a_n) converges to a. □

9.22 Proposition (Compactness Theorem, Second Version). *Let $\mathfrak{C}$ be a uniform class of normed space L-structures, and suppose that Γ is a set of positive bounded L-sentences such that for each finite subset Σ of Γ^+ there exists a structure in $\mathfrak{C}$ which satisfies Σ. Then there exists an ultraproduct of structures from $\mathfrak{C}$ which satisfies (not just approximately satisfies) the entire set Γ.*

Proof. The ultrafilter D used in the proof of Proposition 9.5 is countably incomplete; use Propositions 9.18 and 9.20. □

Existence of Saturated Structures

In the following definition we borrow a term from nonstandard analysis. (See [HL85].)

9.23 Definition. Let $\mathcal{M}, \mathcal{N}$ be normed space L-structures with $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$. We say that $\mathcal{N}$ is an *enlargement of* $\mathcal{M}$ if the following condition holds: whenever C is a set of constants and $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$, and $\Gamma(x_1, \dots, x_n)$ is a set of positive bounded $L(C)$-formulas containing the formulas $\|x_k\| \leq r_k$ for $k = 1, \dots, n$ (where $r_1, \dots, r_n$ are nonnegative rational numbers), if every finite subset of Γ^+ is satisfiable in $(\mathcal{M}, \mathbf{A})$, then the entire set Γ is approximately satisfiable in $(\mathcal{N}, \mathbf{A})$.

Suppose κ is a cardinal such that $\operatorname{card}(L) \leq \kappa$ and $\operatorname{card}(M^{(s)}) \leq \kappa$ for every sort s of L. Note that in Definition 9.23 it suffices only to consider sets of constants C with $\operatorname{card}(C) \leq \kappa$.

It is easy to construct ultrapower enlargements of a given normed space structure, as the following argument shows.

9.24 Lemma. *Let $\mathcal{M}$ be a normed space L-structure, and let κ be an infinite cardinal such that $\operatorname{card}(L) \leq \kappa$ and $\operatorname{card}(M^{(s)}) \leq \kappa$ for every sort s of L. Let I be the collection of finite subsets of κ. There is a countably incomplete ultrafilter D on I such that $(\mathcal{M})_D$ is an enlargement of $\mathcal{M}$.*

Proof. For each ordinal $\alpha < \kappa$, let $S_\alpha = \{i \in I \mid \alpha \in i\}$. Note that the family $\{S_\alpha \mid \alpha < \kappa\}$ has the finite intersection property, so it is contained in an ultrafilter D on I. (Every such D is countably incomplete.) We will show that $(\mathcal{M})_D$ is an enlargement of $\mathcal{M}$.

Let C be a set of new constants with $\operatorname{card}(C) \leq \kappa$, and suppose $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$. Further, let $\Gamma(x_1, \dots, x_n)$ be a set of positive bounded $L(C)$-formulas containing the formulas $\|x_k\| \leq r_k$ for $k = 1, \dots, n$ (where $r_1, \dots, r_n$ are nonnegative rational numbers) such that every finite subset of Γ^+ is satisfiable in $(\mathcal{M}, \mathbf{A})$. We must show that Γ is approximately satisfiable in $((\mathcal{M})_D, \mathbf{A})$. Our assumptions on L, C, κ ensure that there is a surjection from κ onto Γ^+; let the value of this surjection at $\alpha < \kappa$ be φ_α. We define functions $a_1, \dots, a_n$ from I into suitable sorts of $\mathcal{M}$ such that for each $i \in I$ the n-tuple $(a_1(i), \dots, a_n(i))$ satisfies the formulas $\varphi_\alpha(x_1, \dots, x_n)$ in $(\mathcal{M}, \mathbf{A})$ for each $\alpha \in i$ and satisfies the norm bounds $\|a_j(i)\| \leq r_j + 1$ for $j = 1, \dots, n$. Using Proposition 9.2 it follows that $((a_1)_D, \dots, (a_n)_D)$ satisfies $\Gamma^+(x_1, \dots, x_n)$ in $((\mathcal{M})_D, \mathbf{A})$. $\square$

9.25 Proposition. *Let $\mathcal{M}$ be a normed space L-structure. For every* *infinite cardinal κ, $\mathcal{M}$ has a κ-saturated approximate elementary extension.*

Proof. By increasing κ if necessary (for example, replacing κ by κ^+) we may assume κ is regular. By induction we construct an approximate elementary chain $(\mathcal{M}_\alpha \mid \alpha < \kappa)$ such that $\mathcal{M}_0 = \mathcal{M}$ and for each $\alpha < \kappa$, $\mathcal{M}_{\alpha+1}$ is an enlargement of $\mathcal{M}_\alpha$. (At limit ordinals we take unions.) Let $\mathcal{N}$ be the union of the chain $(\mathcal{M}_\alpha \mid \alpha < \kappa)$. By Corollary 9.12, $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$. We claim that $\mathcal{N}$ is κ-saturated. Let C be a set of constants such that $\mathrm{card}(C) < \kappa$ and let $\mathbf{A}$ be any interpretation of C in $\mathcal{N}$. Since κ is regular, there exists $\alpha < \kappa$ such that all the values of $\mathbf{A}$ are elements of $\mathcal{M}_\alpha$. The elements of $\mathcal{N}$ needed to verify Definition 9.17 for $(\mathcal{N}, \mathbf{A})$ can be found in $\mathcal{M}_{\alpha+1}$. □

Satisfaction and Approximate Satisfaction

The results about $\aleph_1$-saturated structures given earlier in this chapter can be used as the basis for an alternative point of view toward defining $\models_{\mathcal{A}}$ between normed space structures and positive bounded formulas. Take any normed space L-structure $\mathcal{M}$, and consider any ultrapower $(\mathcal{M})_D$ of $\mathcal{M}$ with D being a countably incomplete ultrafilter. For example, take D to be any non-principal ultrafilter on a countable index set. The next result shows that approximate satisfaction of positive bounded formulas in $\mathcal{M}$ could be defined using ordinary model theoretic satisfaction of those formulas in $(\mathcal{M})_D$. (In particular, this turns out to be independent of D, as long as D is countably incomplete.)

9.26 Proposition. *Let D be any countably incomplete ultrafilter. If $\mathcal{M}$ is a normed space L-structure, $\varphi(x_1, \dots, x_n)$ is a positive bounded formula, and $a_1, \dots, a_n$ are elements of suitable sorts of $\mathcal{M}$, the following conditions are equivalent:*

(i) $\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \dots, a_n]$;
(ii) $(\mathcal{M})_D \models \varphi[a_1, \dots, a_n]$.

Proof. The implication (ii)⇒(i) is given by Corollary 9.3, and (i)⇒(ii) by Propositions 9.18 and 9.20 (and Corollary 9.3). □

10
Isomorphic Ultrapowers

In this chapter we prove isomorphism theorems for ultrapowers and ultraproducts of normed space structures. These results show that there is a very tight connection between (a) properties that are preserved under the ultraproduct construction and (b) properties that are expressible using the logic for normed space structures that is described in this paper.

Let L be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be two normed space L-structures. If $\mathcal{M}$ and $\mathcal{N}$ have isomorphic ultrapowers, by Corollary 9.4 they must be approximately elementarily equivalent. Theorem 10.7 below gives the converse (in a strong form). Together these results show that ultrapower equivalence of $\mathcal{M}$ and $\mathcal{N}$ is the same as approximate elementary equivalence. (See the discussion of this issue in the Introduction.)

Theorem 10.8 below is a similar result for ultraproducts. Among other things, it shows that the ultrafilter guaranteed by Theorem 10.7 can be chosen in a highly uniform way and that the ultrapowers in question can be taken to be highly saturated. The uniformity will be exploited in Chapter 12 to prove the existence of ultrapowers that are highly homogeneous (in addition to being highly saturated).

The results in this chapter are analogous to the Keisler-Shelah Theorem in ordinary model theory. (See [She71] and Chapter 6 in [CK90].) Moreover, our proof follows a similar line of argument, with adjustments appropriate to the handling of positive bounded formulas and their approximations.

Theorem 10.7 is a special case of Theorem 10.8. We have included both arguments because the proof of Theorem 10.8 is a complicated elaboration of the proof of Theorem 10.7, and because the ultrapower isomorphism result in Theorem 10.7 will be of interest in its own right to many readers.

Combinatorial Lemmas

Throughout this section, λ, τ and κ will denote infinite cardinals with $\lambda = 2^\kappa$ and μ will denote the least cardinal α such that $\lambda^\alpha > \lambda$. Using basic cardinal arithmetic, one can verify that $\kappa < \mu \le \lambda$ and μ is a regular cardinal.

10.1 Definition. Let F be a set of functions from λ into μ and let D be a proper filter on λ. We say that the pair (F, D) is τ*-consistent* if the following two conditions hold:

(1) D is generated by a subset of cardinality at most τ (*i.e.*, there exists a subset E of D of cardinality at most τ such that E is closed under finite intersections and every element of D is a superset of some element of E).

(2) Whenever we are given

- a cardinal $\beta < \mu$,
- a family $(f_i \mid i < \beta)$ of distinct elements of F, and
- a family $(\sigma_i \mid i < \beta)$ of ordinals less than μ,

then D together with

$$\{\xi < \lambda \mid f_i(\xi) = \sigma_i, \text{ for every } i < \beta\}$$

generate a proper filter on λ (*i.e.*, when this set is added to D, the resulting family still has the finite intersection property).

10.2 Remarks.

(1) If (F, D) is τ-consistent, $\tau \le \tau'$, and $F' \subseteq F$, then (F', D) is τ'-consistent.

(2) Suppose that $(\tau_i \mid i < \delta)$ is an increasing family of infinite cardinals such that $\sup_i \tau_i = \tau$. Suppose also that (F_i, D_i) is τ_i-consistent for all $i < \delta$ and that $F_j \supseteq F_i$ and $D_j \subseteq D_i$ for $j < i < \delta$. Then the pair $\left(\bigcap_i F_i,\ \bigcup_i D_i\right)$ is τ-consistent.

10.3 Remark. The pair (F, D) is τ-consistent in the sense of Definition 10.1 if and only if the triple $(F, \emptyset, D)$ is τ-consistent as defined in [She71]; see also page 394 in [CK90].

10.4 Lemma. *There is a set F of functions from λ into μ such that* $\operatorname{card}(F) = 2^\lambda$ *and $(F, \{\lambda\})$ is μ-consistent.*

Proof. See Lemma 1 of [She71] (or Lemma 6.1.10 in [CK90]). $\square$

10.5 Lemma. *Suppose that* $\tau \geq \mu$, (F, D) *is* τ*-consistent and* $(X_i \mid i < \tau)$ *is a family of subsets of* λ. *Then there exist* $F' \subseteq F$ *and* $D' \supseteq D$ *such that* $\operatorname{card}(F \setminus F') \leq \tau$, (F', D') *is* τ*-consistent, and for every* $i < \tau$, *either* $X_i \in D'$ *or* $\lambda \setminus X_i \in D'$.

Proof. See Lemma 4(B) of [She71] (or Lemma 6.1.13(ii) of [CK90]). □

10.6 Lemma. *Suppose that the following conditions hold:*

- L *is a signature and* x *is a variable in* L *of sort* s,
- $\mathcal{M}$ *is a normed space* L*-structure,*
- M *is a dense subset of the sort of* $\mathcal{M}$ *indexed by* s *and* $\operatorname{card}(M) < \mu$,
- (F, D) *is* τ*-consistent,*
- $(\varphi_i(x, y_{i,1}, \ldots, y_{i,n(i)}) \mid i < \tau)$ *is a family of positive bounded formulas that is closed under conjunctions, and*
- *for each* $i < \tau$, $\bar{a}_i$ *is a function on* λ *such that for all* $\xi < \lambda$, $\bar{a}_i(\xi) = (a_{i,1}(\xi), \ldots, a_{i,n(i)}(\xi))$, *where* $a_{i,1}(\xi), \ldots, a_{i,n(i)}(\xi)$ *are elements of suitable sorts of* $\mathcal{M}$, *and*

$$\{\, \xi < \lambda \mid \mathcal{M} \models \exists x \varphi_i(x, y_{i,1}, \ldots, y_{i,n(i)})[a_{i,1}(\xi), \ldots, a_{i,n(i)}(\xi)] \,\} \in D.$$

Then, there exist $F' \subseteq F$ *and* $D' \supseteq D$ *such that* $\operatorname{card}(F \setminus F') \leq \tau$ *and* (F', D') *is* τ*-consistent, and there is a function* $b \colon \lambda \to M$ *such that for every* $i < \tau$ *and every approximation* φ_i' *of* φ_i

$$\{\, \xi < \lambda \mid \mathcal{M} \models \varphi_i'[b(\xi), a_{i,1}(\xi), \ldots, a_{i,n(i)}(\xi)] \,\} \in D'.$$

Proof. Similar to Lemma 5 of [She71] (or Lemma 6.1.14 of [CK90]). The Perturbation Lemma (Proposition 5.15) is used to ensure that the values of the function b come from the dense set M. Note that each positive bounded formula has only countably many approximations. □

The Ultrapower Theorem

10.7 Theorem. *Let* L *be a signature and suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are normed space* L*-structures which satisfy* $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$. *Then there exists an ultrafilter* D *such that* $(\mathcal{M})_D \cong (\mathcal{N})_D$.

Proof. Note that this result is the converse of Corollary 9.4.

Let κ be an infinite cardinal number such that the density character of each sort of $\mathcal{M}$ and $\mathcal{N}$ is at most κ and such that $\operatorname{card}(L) \leq 2^\kappa$.

In order to simplify notation, in the proof of Theorem 10.7 we will assume that $\mathcal{M}$ and $\mathcal{N}$ each have only one sort (in addition to the special sort $\mathbb{R}$) and for each structure we will refer to this sort as *the universe*.

Let M, N be dense subsets of the universes of $\mathcal{M}, \mathcal{N}$ respectively such that $\operatorname{card}(M)$, $\operatorname{card}(N)$ are both $\leq \kappa$.

Let $\lambda = 2^\kappa$ and let μ be the least cardinal α such that $\lambda^\mu > \lambda$. Note that $\kappa < \mu \leq \lambda$ and μ is regular. We use an inductive construction to find simultaneously an ultrafilter D on λ and an isomorphism between $(\mathcal{M})_D$ and $(\mathcal{N})_D$.

Consider a family $a = (a(\xi) \mid \xi < \lambda)$ of elements of M; we say that the family is *bounded* if the norms $(\|a(\xi)\| \mid \xi < \lambda)$ are uniformly bounded. We let $\ell_\infty(\lambda, M)$ denote the set of all such bounded families from M and define $\|a\|_\infty$ to be the supremum of all $\|a(\xi)\|$ for $\xi < \lambda$. We make a similar definition for N in place of M. Note that the cardinalities of $\ell_\infty(\lambda, M)$ and $\ell_\infty(\lambda, N)$ are at most 2^λ. We fix listings of the elements of $\ell_\infty(\lambda, M)$ and $\ell_\infty(\lambda, N)$ indexed over the ordinals $< 2^\lambda$.

We use Lemma 10.4 to fix a set F_0 of functions from λ into μ such that $\operatorname{card}(F_0) = 2^\lambda$ and $(F_0, \{\lambda\})$ is μ-consistent (and hence it is λ-consistent). Let $D_0 = \{\lambda\}$. Let $(S_j \mid j < 2^\lambda)$ be a list of all the subsets of λ.

By induction on all ordinals $i < 2^\lambda$, we construct

- a decreasing family $(F_i \mid i < 2^\lambda)$ of sets of functions from λ into μ,
- an increasing family $(D_i \mid i < 2^\lambda)$ of proper filters on λ, and
- families $(a_i \mid i < 2^\lambda)$ and $(b_i \mid i < 2^\lambda)$ in $\ell_\infty(\lambda, M)$ and $\ell_\infty(\lambda, N)$, respectively,

such that the following conditions hold for all $i < 2^\lambda$.

(1)(a) $\operatorname{card}(F_0 \setminus F_i) \leq \lambda + \operatorname{card}(i)$ (and therefore $\operatorname{card}(F_i) = 2^\lambda$);
(b) (F_i, D_i) is $(\lambda + \operatorname{card}(i))$-consistent.

(2) If $i = j + 1$, then either $S_j \in D_i$ or $\lambda \setminus S_j \in D_i$.

(3) If $i = j + 1$ and j is an even ordinal, then there exists a filter D'_j on λ such that

$$D_j \subseteq D'_j \subseteq D_i,$$

and for every positive bounded L-formula $\varphi(x, x_1, \ldots, x_n)$ and every choice of $j_1, \ldots, j_n < j$:

(a) either the set

$$\{\, \xi < \lambda \mid \mathcal{M} \models \varphi[a_j(\xi), a_{j_1}(\xi), \ldots, a_{j_n}(\xi)] \,\}$$

or its complement is in D'_j;

(b) for every $\psi > \varphi$,

$$\{\,\xi < \lambda \mid \mathcal{M} \models \varphi[a_j(\xi), a_{j_1}(\xi), \ldots, a_{j_n}(\xi)]\,\} \in D'_j$$

implies

$$\{\,\xi < \lambda \mid \mathcal{N} \models \psi[b_j(\xi), b_{j_1}(\xi), \ldots, b_{j_n}(\xi)]\,\} \in D_i.$$

(4) If $i = j + 1$ and j is an odd ordinal, then there exists a filter D'_j on λ such that

$$D_j \subseteq D'_j \subseteq D_i,$$

and for every positive bounded L-formula $\varphi(x, x_1, \ldots, x_n)$ and every choice of $j_1, \ldots, j_n < j$:

(a) either the set

$$\{\,\xi < \lambda \mid \mathcal{N} \models \varphi[b_j(\xi), b_{j_1}(\xi), \ldots, b_{j_n}(\xi)]\,\}$$

or its complement is in D'_j;

(b) for every $\psi > \varphi$,

$$\{\,\xi < \lambda \mid \mathcal{N} \models \varphi[b_j(\xi), b_{j_1}(\xi), \ldots, b_{j_n}(\xi)]\,\} \in D'_j$$

implies

$$\{\,\xi < \lambda \mid \mathcal{M} \models \psi[a_j(\xi), a_{j_1}(\xi), \ldots, a_{j_n}(\xi)]\,\} \in D_i.$$

(5) If i is a limit ordinal,

$$F_i = \bigcap_{j<i} F_j \qquad D_i = \bigcup_{j<i} D_j.$$

After this construction is completed, we will let $D = \bigcup_{i<2^\lambda} D_i$. Condition (2) ensures that D is an ultrafilter. The inductive construction will be carried out so that the families $(a_i \mid i < 2^\lambda)$ and $(b_i \mid i < 2^\lambda)$ exhaust $\ell_\infty(\lambda, M)$ and $\ell_\infty(\lambda, N)$ respectively. Therefore the family of equivalence classes $((a_i)_D \mid i < 2^\lambda)$ is norm dense in $(\mathcal{M})_D$ and $((b_i)_D \mid i < 2^\lambda)$ is dense in $(\mathcal{N})_D$.

The desired isomorphism T of $(\mathcal{M})_D$ onto $(\mathcal{N})_D$ is initially defined to take $(a_i)_D$ to $(b_i)_D$ for each $i < 2^\lambda$. To show that T has the desired properties, fix $\varphi(x_1, \ldots, x_n)$ and suppose

$$\{\,\xi < \lambda \mid \mathcal{M} \models \varphi[a_{j_1}(\xi), \ldots, a_{j_n}(\xi)]\,\} \in D,$$

where $j_1, \ldots, j_n < 2^\lambda$. Let j be the least even ordinal such that

$$j_1, \ldots, j_n < j.$$

By (3)(a) we have

$$\{\, \xi < \lambda \mid \mathcal{M} \models \varphi[a_{j_1}(\xi), \dots, a_{j_n}(\xi)] \,\} \in D'_j,$$

so by (3)(b) for every $\psi > \varphi$ we have

$$\{\, \xi < \lambda \mid \mathcal{N} \models \psi[b_{j_1}(\xi), \dots, b_{j_n}(\xi)] \,\} \in D_{j+1} \subseteq D.$$

It follows that T extends to an isomorphism from $(\mathcal{M})_D$ onto $(\mathcal{N})_D$.

We now describe the construction of $(F_i \mid i < 2^\lambda)$, $(D_i \mid i < 2^\lambda)$, $(a_i \mid i < 2^\lambda)$, and $(b_i \mid i < 2^\lambda)$. If i is a limit ordinal, we define F_i and D_i to make clause (5) true and use Remark 10.2(2) to verify clause (1)(b). All other conditions to be satisfied apply to the case where i is a successor ordinal, say $i = j + 1$. The order in which we define the next elements of our four families is: a_j, b_j and then D_i, F_i.

If j is even, we define a_j to be the first element of $\ell_\infty(\lambda, M)$ that does not appear in the list $(a_l \mid l < j)$ and then construct b_j and D_i, F_i in that order; if j is odd we define b_j to be the first element of $\ell_\infty(\lambda, N)$ that does not appear in the list $(b_l \mid l < j)$ and then construct a_j and D_i, F_i in that order. Because of the symmetry between these situations, we will discuss only the case when j is even.

For each positive bounded L-formula $\varphi(x, y_1, \dots, y_n)$ and each finite sequence of ordinals $j_1, \dots, j_n < j$, let

$$X(\varphi, j_1, \dots, j_n) = \{\, \xi < \lambda \mid \mathcal{M} \models \varphi[a_j(\xi), a_{j_1}(\xi), \dots, a_{j_n}(\xi)] \,\}.$$

There are at most $\lambda + \mathrm{card}(i)$ such sets X. Hence, since (F_j, D_j) is $(\lambda + \mathrm{card}(i))$-consistent, Lemma 10.5 provides $F'_j \subseteq F_j$ and $D'_j \supseteq D_j$ such that

- $\mathrm{card}(F_j \setminus F'_j) \le \lambda + \mathrm{card}(i)$,
- (F'_j, D'_j) is $(\lambda + \mathrm{card}(i))$-consistent,
- for every choice of $\varphi(x, y_1, \dots, y_n)$ and $j_1, \dots, j_n < j$,

$$X(\varphi, j_1, \dots, j_n) \in D'_j \quad \text{or} \quad \lambda \setminus X(\varphi, j_1, \dots, j_n) \in D'_j,$$

- either $S_j \in D'_j$ or $\lambda \setminus S_j \in D'_j$.

Fix a formula $\varphi(x, y_1, \dots, y_n)$ and $j_1, \dots, j_n < j$, and suppose that $X(\varphi, j_1, \dots, j_n) \in D'_j$. Let u be a rational number such that $u > \|a_j\|_\infty$. Then the set $Y(u, \varphi, j_1, \dots, j_n)$ is in D'_j, where

$$Y(u, \varphi, j_1, \dots, j_n) = \{\, \xi < \lambda \mid \mathcal{M} \models \exists_u x\ \varphi(x, y_1, \dots, y_n)[a_{j_1}(\xi), \dots, a_{j_n}(\xi)] \,\}.$$

We claim that for every rational number $w > u$ and every approximation ψ of φ, the set $Z(w, \psi, j_1, \ldots, j_n)$ is also in D'_j, where

$$Z(w, \psi, j_1, \ldots, j_n) = \{\, \xi < \lambda \mid \mathcal{N} \models \exists_w x\ \psi(x, y_1, \ldots, y_n)[b_{j_1}(\xi), \ldots, b_{j_n}(\xi)] \,\}.$$

To see this, suppose $Z(w, \psi, j_1, \ldots, j_n) \notin D'_j$. Then let $l = \max_k j_k$, and fix an approximation φ' of φ such that $\varphi < \varphi' < \psi$ and a rational number v such that $u < v < w$. We have $Z(w, \psi, j_1, \ldots, j_n) \notin D_l$ and hence $\lambda \setminus Z(w, \psi, j_1, \ldots, j_n) \in D_l$ by (4)(a). But then,

$$\{\, \xi < \lambda \mid \mathcal{N} \models \forall_w x\ \ \mathrm{neg}(\psi)(x, y_1, \ldots, y_n)[b_{j_1}(\xi), \ldots, b_{j_n}(\xi)] \,\} \in D_l,$$

so by (4)(b),

$$\{\, \xi < \lambda \mid \mathcal{M} \models \forall_v x\ \ \mathrm{neg}(\varphi')(x, y_1, \ldots, y_n)[a_{j_1}(\xi), \ldots, a_{j_n}(\xi)] \,\} \in D_j \subseteq D'_j,$$

which contradicts the fact that D'_j is a proper filter.

There are two situations in which the description above needs to be clarified, at the point where we are showing that $Z(w, \psi, j_1, \ldots, j_n) \notin D'_j$ and we set $l = \max_k j_k$. When $j = 0$ we do not have $j_1, \ldots, j_n$ and the formulas involved have no free variables; here we use the hypothesis that $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$ and recall that $D_0 = \{\lambda\}$. When $j > 0$ we may assume $n \geq 1$ without loss of generality and the argument proceeds as given.

We now apply Lemma 10.6 to obtain $F_i \subseteq F'_j$ and $D_i \supseteq D'_j$ such that $\mathrm{card}(F'_j \setminus F_i) \leq \lambda + \mathrm{card}(i)$ and (F_i, D_i) is $(\lambda + \mathrm{card}(i))$-consistent, and a function $b_j \colon \lambda \to N$ such that

$$\{\, \xi < \lambda \mid \mathcal{N} \models \psi[b_j(\xi), b_{j_1}(\xi), \ldots, b_{j_n}(\xi)] \,\} \in D_i$$

for every positive bounded formula $\varphi(x, y_1, \ldots, y_n)$ such that

$$X(\varphi, j_1, \ldots, j_n) \in D'_j$$

and every approximation ψ of φ. In particular, we can assume $\|b_j\|_\infty \leq \|a_j\|_\infty$, so b_j is bounded.

Conditions (1)–(4) are immediate from the construction. □

The Ultraproduct Theorem

10.8 Theorem. *Let κ be an infinite cardinal and let $\lambda = 2^\kappa$. Then there exists an ultrafilter D on λ satisfying the following properties: if L is a signature with* $\mathrm{card}(L) \leq \lambda$ *and* $(\mathcal{M}_\xi \mid \xi < \lambda)$ *and* $(\mathcal{N}_\xi \mid \xi < \lambda)$ *are* *families of normed space L-structures such that the density character of each sort of $\mathcal{M}_\xi$ and $\mathcal{N}_\xi$ is at most κ for all $\xi < \lambda$, then*

$$(\prod_{\xi<\lambda} \mathcal{M}_\xi)_D \equiv_{\mathcal{A}} (\prod_{\xi<\lambda} \mathcal{N}_\xi)_D$$

implies

$$(\prod_{\xi<\lambda} \mathcal{M}_\xi)_D \cong (\prod_{\xi<\lambda} \mathcal{N}_\xi)_D.$$

Moreover, in each such situation the ultraproduct $(\prod_{\xi<\lambda} \mathcal{M}_\xi)_D$ is κ^+-saturated.

Proof. Let μ be the smallest cardinal number α such that $\lambda^\alpha > \lambda$, so $\kappa < \mu \leq \lambda$ and μ is regular.

To clarify the proof, we have divided it into five parts.

PART 1. SOME TECHNICAL SIMPLIFICATIONS. In order to simplify notation, we will explicitly treat only normed space structures which have only one sort in addition to the special sort $\mathbb{R}$. The calligraphic letters $\mathcal{M}$, $\mathcal{N}$, etc. will be used to denote normed space structures, and the corresponding Roman letters M, N, etc. will denote dense subsets of their respective universes with cardinality at most κ.

Without loss of generality, we fix a signature L of cardinality λ such that every signature of cardinality at most λ occurs as a subsignature of L. We also restrict our attention to structures $\mathcal{M}$ whose universe has a dense subset M that is a cardinal $\leq \kappa$ and whose universe is a cardinal $\leq \lambda$. Each normed space structure needing to be considered is isomorphic to one satisfying these restrictions. The number of subsignatures of L is 2^λ, and for each such subsignature L_0 and each $\chi \leq \kappa$, the number of L_0-structures whose universe is a cardinal $\leq \lambda$ that has χ as a dense subset is $(\lambda^\chi)^{\mathrm{card}(L_0)} \leq \lambda^\lambda = 2^\lambda$. Hence the number of normed space structures under consideration is 2^λ.

PART 2. A LIST OF STRUCTURES, FILTERS, AND ELEMENTS. Let

$$(G_\alpha \mid \alpha < 2^\lambda)$$

be a list of all the filters on λ which have a set of at most λ generators.

For each $\alpha < 2^\lambda$ let

$$(\, (\mathcal{M}_\xi^{\alpha,\gamma}, \mathcal{N}_\xi^{\alpha,\gamma})_{\xi<\lambda} \mid \gamma < 2^\lambda \,)$$

be a list of all the families $(\mathcal{M}_\xi^{\alpha,\gamma}, \mathcal{N}_\xi^{\alpha,\gamma})_{\xi<\lambda}$ which satisfy the following conditions:

- There exists a subsignature L^α of L such that for every $\xi < \lambda$, $\mathcal{M}_\xi^{\alpha,\gamma}$ and $\mathcal{N}_\xi^{\alpha,\gamma}$ are L^α-structures.
- For every $\xi < \lambda$ the universes of $\mathcal{M}_\xi^{\alpha,\gamma}$ and $\mathcal{N}_\xi^{\alpha,\gamma}$ are cardinals $\leq \lambda$ and have dense subsets that are cardinals $\leq \kappa$.
- For every L^α-sentence θ and every $\theta' > \theta$, either the set
$$\{\, \xi < \lambda \mid \mathcal{M}_\xi^{\alpha,\gamma} \models \theta' \,\}$$
or its complement is in G_α.
- For every L^α-sentence θ and every $\theta' > \theta$, either the set
$$\{\, \xi < \lambda \mid \mathcal{N}_\xi^{\alpha,\gamma} \models \theta' \,\}$$
or its complement is in G_α.
- For every L^α-sentence θ and every $\theta' > \theta$,
$$\{\xi < \lambda \mid \mathcal{M}_\xi^{\alpha,\gamma} \models \theta\} \in G_\alpha \quad \text{implies} \quad \{\xi < \lambda \mid \mathcal{N}_\xi^{\alpha,\gamma} \models \theta'\} \in G_\alpha$$
and
$$\{\xi < \lambda \mid \mathcal{N}_\xi^{\alpha,\gamma} \models \theta\} \in G_\alpha \quad \text{implies} \quad \{\xi < \lambda \mid \mathcal{M}_\xi^{\alpha,\gamma} \models \theta'\} \in G_\alpha.$$

For each $\alpha, \gamma < 2^\lambda$ and $\xi < \lambda$ we let $M_\xi^{\alpha,\gamma}, N_\xi^{\alpha,\gamma}$ denote the cardinals $\leq \kappa$ that are dense subsets of the universes of $\mathcal{M}_\xi^{\alpha,\gamma}, \mathcal{N}_\xi^{\alpha,\gamma}$ respectively.

For $\alpha, \gamma < 2^\lambda$, let
$$\{\, a_\rho^{\alpha,\gamma} \mid \rho < 2^\lambda \,\}$$
be the set of bounded elements of $\prod_{\xi<\lambda} M_\xi^{\alpha,\gamma}$ and
$$\{\, b_\rho^{\alpha,\gamma} \mid \rho < 2^\lambda \,\}$$
the set of bounded elements of $\prod_{\xi<\lambda} N_\xi^{\alpha,\gamma}$.

Fix a family
$$C = (c_i \mid i < 2^\lambda)$$
of distinct constant symbols which do not occur in L, and for each $i < 2^\lambda$ let
$$C_i = \{\, c_j \mid j < i \,\}.$$

Using cardinality considerations, we can fix a function g defined on the set $\{i \mid i < 2^\lambda\}$ satisfying the properties listed below.

(I) For each $i < 2^\lambda$, one of the following conditions holds:

(i) $g(i) = (0, S)$, where S is a subset of λ.

(ii) $g(i) = (1, \alpha, \gamma, \rho, a_\rho^{\alpha,\gamma})$, where $\alpha, \gamma, \rho < 2^\lambda$;

(iii) $g(i) = (2, \alpha, \gamma, \rho, b_\rho^{\alpha,\gamma})$, where $\alpha, \gamma, \rho < 2^\lambda$;

(iv) $g(i) = (3, \alpha, \gamma, K, p, R)$, where K is a subset of C_i with $\mathrm{card}(K) \leq \kappa$, $p = p(x)$ is a set of positive bounded $L(K)$-formulas which is closed under conjunction, and R is a nonnegative rational number;

(v) $g(i) = (4, \alpha, \gamma, K, p, R)$, where K is a subset of C_i with $\mathrm{card}(K) \leq \kappa$, $p = p(x)$ is a set of positive bounded $L(K)$-formulas which is closed under conjunction, and R is a nonnegative rational number.

(II) Each of the following elements can be expressed as $g(i)$ for a unique $i < 2^\lambda$:

(i) Every pair of the form $(0, S)$, where S is a subset of λ;

(ii) Every quintuple of the form $(1, \alpha, \gamma, \rho, a_\rho^{\alpha,\gamma})$, where $\alpha, \gamma, \rho < 2^\lambda$;

(iii) Every quintuple of the form $(2, \alpha, \gamma, \rho, b_\rho^{\alpha,\gamma})$, where $\alpha, \gamma, \rho < 2^\lambda$;

(iv) Every sextuple of the form $(3, \alpha, \gamma, K, p, R)$, where K is a subset of C_i with $\mathrm{card}(K) \leq \kappa$, $p = p(x)$ is a set of positive bounded $L(K)$-formulas which is closed under conjunction, and R is a nonnegative rational number.

(v) Every sextuple of the form $(4, \alpha, \gamma, K, p, R)$, where K is a subset of C_i with $\mathrm{card}(K) \leq \kappa$, $p = p(x)$ is a set of positive bounded $L(K)$-formulas which is closed under conjunction, and R is a nonnegative rational number.

(III) For fixed α and γ, the cases given by (I)(ii) and (I)(iii) above alternate; more precisely,

(i) If $g(i) = (1, \alpha, \gamma, \rho, a_\rho^{\alpha,\gamma})$ and

$$\begin{aligned} i^* = \min_{j>i} \Big[&\text{there exists } \sigma < 2^\lambda \text{ such that} \\ &g(j) = (1, \alpha, \gamma, \sigma, a_\sigma^{\alpha,\gamma}) \quad \text{or} \\ &g(j) = (2, \alpha, \gamma, \sigma, b_\sigma^{\alpha,\gamma}) \Big], \end{aligned}$$

then $g(i^*) = (2, \alpha, \gamma, \sigma, b_\sigma^{\alpha,\gamma})$ for some $\sigma < 2^\lambda$.

(ii) If $g(i) = (2, \alpha, \gamma, \rho, b_\rho^{\alpha,\gamma})$ and

$$i^* = \min_{j>i} \Big[\text{there exists } \sigma < 2^\lambda \text{ such that } g(j) = (1, \alpha, \gamma, \sigma, a_\sigma^{\alpha,\gamma}) \quad \text{or} \quad g(j) = (2, \alpha, \gamma, \sigma, b_\sigma^{\alpha,\gamma})\Big],$$

then $g(i^*) = (1, \alpha, \gamma, \sigma, a_\sigma^{\alpha,\gamma})$ for some $\sigma < 2^\lambda$.

PART 3. AN INDUCTIVE CONSTRUCTION. The construction is an elaboration of the ideas used in the proof of Theorem 10.7, with which we assume the reader is familiar. First, we use Lemma 10.4 to fix a set F_0 of 2^λ functions from λ into μ such that $(F_0, \{\lambda\})$ is μ-consistent (and hence λ-consistent). Then we let $D_0 = \{\lambda\}$, and for $\alpha, \gamma < 2^\lambda$ we let $A_0^{\alpha,\gamma}$ and $B_0^{\alpha,\gamma}$ be the empty functions.

By induction on all ordinals $i < 2^\lambda$, we will now construct:

- a decreasing family $(F_i \mid i < 2^\lambda)$ of sets of functions from λ into μ,
- an increasing family $(D_i \mid i < 2^\lambda)$ of proper filters on λ,
- for each choice of $\alpha, \gamma < 2^\lambda$ functions

$$A_i^{\alpha,\gamma} \colon C_i \to \ell_\infty(\,\lambda, (M_\xi^{\alpha,\gamma} \mid \xi < \lambda)\,),$$
$$B_i^{\alpha,\gamma} \colon C_i \to \ell_\infty(\,\lambda, (N_\xi^{\alpha,\gamma} \mid \xi < \lambda)\,),$$

where $A_i^{\alpha,\gamma}$ extends $A_j^{\alpha,\gamma}$ for $i > j$ and $B_i^{\alpha,\gamma}$ extends $B_j^{\alpha,\gamma}$ for $i > j$,

such that the following conditions hold:

(1)(a) $\mathrm{card}(F_0 \setminus F_i) \le \lambda + \mathrm{card}(i)$,

(b) (F_i, D_i) is $(\lambda + \mathrm{card}(i))$-consistent.

(2) If $i = j + 1$ and $g(j) = (0, S)$, then either $S \in D_i$ or $\lambda \setminus S \in D_i$.

(3) If $i = j + 1$, $g(j) = (1, \alpha, \gamma, \rho, a_\rho^{\alpha,\gamma})$, and the set $D_j \cup G_\alpha$ generates a proper filter on λ, then there exist a filter D_j' with

$$D_j \subseteq D_j' \subseteq D_i$$

and a bounded element

$$b \in \prod_{\xi<\lambda} N_\xi^{\alpha,\gamma}$$

such the functions $A_i^{\alpha,\gamma}$,$B_i^{\alpha,\gamma}$ are defined by

$$A_i^{\alpha,\gamma}(c) = \begin{cases} A_j^{\alpha,\gamma}(c), & \text{if } c \in C_j \\ a_\rho^{\alpha,\gamma}, & \text{if } c = c_j, \end{cases}$$

$$B_i^{\alpha,\gamma}(c) = \begin{cases} B_j^{\alpha,\gamma}(c), & \text{if } c \in C_j \\ b, & \text{if } c = c_j, \end{cases}$$

and the following conditions are satisfied: given a positive bounded formula $\varphi(x, x_1, \ldots, x_n)$ and ordinals $j_1, \ldots, j_n < j$, if

$$A_j^{\alpha,\gamma}(c_{j_k}) = a_k \qquad (k = 1, \ldots, n),$$
$$B_j^{\alpha,\gamma}(c_{j_k}) = b_k \qquad (k = 1, \ldots, n),$$

then:

(a) either the set

$$\left\{\ \xi < \lambda \ \middle|\ \mathcal{M}_\xi^{\alpha,\gamma} \models \varphi[a_\rho^{\alpha,\gamma}(\xi), a_1(\xi), \ldots, a_n(\xi)]\ \right\}$$

or its complement is in D'_j; and

(b) for every approximation φ' of φ,

$$\left\{\ \xi < \lambda \ \middle|\ \mathcal{M}_\xi^{\alpha,\gamma} \models \varphi[a_\rho^{\alpha,\gamma}(\xi), a_1(\xi), \ldots, a_n(\xi)]\ \right\} \in D'_j$$

implies

$$\left\{\ \xi < \lambda \ \middle|\ \mathcal{N}_\xi^{\alpha,\gamma} \models \varphi'[b(\xi), b_1(\xi), \ldots, b_n(\xi)]\ \right\} \in D_i.$$

(4) If $i = j + 1$, $g(j) = (2, \alpha, \gamma, \rho, b_\rho^{\alpha,\gamma})$, and the set $D_j \cup G_\alpha$ generates a proper filter on λ, then there exist a filter D'_j with

$$D_j \subseteq D'_j \subseteq D_i$$

and a bounded element

$$a \in \prod_{\xi<\lambda} M_\xi^{\alpha,\gamma}$$

such that the functions $A_i^{\alpha,\gamma}$,$B_i^{\alpha,\gamma}$ are defined by

$$A_i^{\alpha,\gamma}(c) = \begin{cases} A_j^{\alpha,\gamma}(c), & \text{if } c \in C_j \\ a, & \text{if } c = c_j, \end{cases}$$

$$B_i^{\alpha,\gamma}(c) = \begin{cases} B_j^{\alpha,\gamma}(c), & \text{if } c \in C_j \\ b_\rho^{\alpha,\gamma}, & \text{if } c = c_j, \end{cases}$$

and the following conditions are satisfied: given a positive bounded formula $\varphi(x, x_1, \ldots, x_n)$ and ordinals $j_1, \ldots, j_n < j$, if

$$A_j^{\alpha,\gamma}(c_{j_k}) = a_k \qquad (k = 1, \ldots, n),$$
$$B_j^{\alpha,\gamma}(c_{j_k}) = b_k \qquad (k = 1, \ldots, n),$$

then:

(a) either the set

$$\left\{ \xi < \lambda \;\middle|\; \mathcal{N}_\xi^{\alpha,\gamma} \models \varphi[b_\rho^{\alpha,\gamma}(\xi), b_1(\xi), \ldots, b_n(\xi)] \right\}$$

or its complement is in D'_j; and

(b) for every approximation φ' of φ,

$$\left\{ \xi < \lambda \;\middle|\; \mathcal{N}_\xi^{\alpha,\gamma} \models \varphi[b_\rho^{\alpha,\gamma}(\xi), b_1(\xi), \ldots, b_n(\xi)] \right\} \in D'_j$$

implies

$$\left\{ \xi < \lambda \;\middle|\; \mathcal{M}_\xi^{\alpha,\gamma} \models \varphi'[a(\xi), a_1(\xi), \ldots, a_n(\xi)] \right\} \in D_i.$$

(5) If $i = j + 1$, $g(j) = (3, \alpha, \gamma, K, p, R)$, $K \subseteq \operatorname{dom}(A_j^{\alpha,\gamma})$, and for every finite subset $p_0(x)$ of $(p(x))^+$ we have

$$\left\{ \xi < \lambda \;\middle|\; (\mathcal{M}_\xi^{\alpha,\gamma}, A_j^{\alpha,\gamma}(c)(\xi) \mid c \in K) \models \exists_R x \bigwedge p_0(x) \right\} \in D_j,$$

then there exists a bounded element a of $\prod_{\xi<\lambda} M_\xi^{\alpha,\gamma}$ such that

$$\left\{ \xi < \lambda \;\middle|\; (\mathcal{M}_\xi^{\alpha,\gamma}, A_j^{\alpha,\gamma}(c)(\xi) \mid c \in K) \models p[a] \right\} \in D_i.$$

(6) If $i = j + 1$, $g(j) = (4, \alpha, \gamma, K, p, R)$, $K \subseteq \operatorname{dom}(B_j^{\alpha,\gamma})$, and for every finite subset $p_0(x)$ of $(p(x))^+$ we have

$$\left\{ \xi < \lambda \;\middle|\; (\mathcal{N}_\xi^{\alpha,\gamma}, B_j^{\alpha,\gamma}(c)(\xi) \mid c \in K) \models \exists_R x \bigwedge p_0(x) \right\} \in D_j,$$

then there exists a bounded element b of $\prod_{\xi<\lambda} N_\xi^{\alpha,\gamma}$ such that

$$\left\{ \xi < \lambda \;\middle|\; (\mathcal{N}_\xi^{\alpha,\gamma}, B_j^{\alpha,\gamma}(c)(\xi) \mid c \in K) \models p[b] \right\} \in D_i.$$

(7) If i is a limit ordinal, then

- $F_i = \bigcap_{j<i} F_j$,
- $D_i = \bigcup_{j<i} D_j$,
- $A_i^{\alpha,\gamma} = \bigcup_{j<i} A_j^{\alpha,\gamma}$,
- $B_i^{\alpha,\gamma} = \bigcup_{j<i} B_j^{\alpha,\gamma}$.

PART 4. DESCRIPTION OF THE INDUCTIVE CONSTRUCTION. As in the proof of Theorem 10.7, the case that requires special attention is the successor case, $i = j + 1$.

If $g(j) = (0, S)$, we use Lemma 10.5 to find F_i and D_i as stated in (2), and define

$$A_i^{\alpha,\gamma}(c) = \begin{cases} A_j^{\alpha,\gamma}(c), & \text{if } c \in C_j \\ 0, & \text{if } c = c_j, \end{cases}$$

$$B_i^{\alpha,\gamma}(c) = \begin{cases} B_j^{\alpha,\gamma}(c), & \text{if } c \in C_j \\ 0, & \text{if } c = c_j. \end{cases}$$

If $g(j) = (1, \alpha, \gamma, \rho, a_\rho^{\alpha,\gamma})$ and the set $D_j \cup G_\alpha$ generates a proper filter on λ, we define $A_i^{\alpha,\gamma}$ and $B_i^{\alpha,\gamma}$ as given in (3) and proceed as in the proof of Theorem 10.7 to find F_i and D_i as stated in (3).

If $g(j) = (2, \alpha, \gamma, \rho, b_\rho^{\alpha,\gamma})$ and the set $D_j \cup G_\alpha$ generates a proper filter on λ, we define $A_i^{\alpha,\gamma}$ and $B_i^{\alpha,\gamma}$ as given in (4) and proceed as in the proof of Theorem 10.7 to find F_i and D_i as stated in (4).

If $i = j + 1$, $g(j) = (3, \alpha, \gamma, K, p, R)$, $K \subseteq \mathrm{dom}(A_j^{\alpha,\gamma})$, and for every finite subset $p_0(x)$ of $(p(x))^+$ we have

$$\left\{ \xi < \lambda \;\middle|\; (\mathcal{M}_\xi^{\alpha,\gamma}, A_j^{\alpha,\gamma}(c)(\xi) \mid c \in K) \models \exists_R x \bigwedge p_0(x) \right\} \in D_j,$$

we use Lemma 10.6 to find F_i and D_i as stated in (5), and let $A_i^{\alpha,\gamma}, B_i^{\alpha,\gamma}$ be as when $g(j) = (0, S)$.

If $i = j + 1$, $g(j) = (4, \alpha, \gamma, K, p, R)$, $K \subseteq \mathrm{dom}(B_j^{\alpha,\gamma})$, and for every finite subset $p_0(x)$ of $(p(x))^+$ we have

$$\left\{ \xi < \lambda \;\middle|\; (\mathcal{N}_\xi^{\alpha,\gamma}, B_j^{\alpha,\gamma}(c)(\xi) \mid c \in K) \models \exists_R x \bigwedge p_0(x) \right\} \in D_j,$$

we use Lemma 10.6 to find F_i and D_i as stated in (6), and let $A_i^{\alpha,\gamma}, B_i^{\alpha,\gamma}$ be as when $g(j) = (0, S)$.

If $i = j + 1$ and $g(j)$ is not of any of the preceding forms, we simply let $F_i = F_j$, $D_i = D_j$, and let $A_i^{\alpha,\gamma}, B_i^{\alpha,\gamma}$ be as when $g(j) = (0, S)$.

PART 5. HOW THE CONSTRUCTION YIELDS THE THEOREM. Let

$$D = \bigcup_{i < 2^\lambda} D_i.$$

By (2), D is an ultrafilter.

Suppose that L_0 is a subsignature of L and

$$(\mathcal{M}_\xi \mid \xi < \lambda), \qquad (\mathcal{N}_\xi \mid \xi < \lambda)$$

are families of normed space L_0-structures such that the density character of each sort of $\mathcal{M}_\xi$ and $\mathcal{N}_\xi$ is at most κ for all $\xi < \lambda$; suppose further that

$$\Big(\prod_{\xi<\lambda} \mathcal{M}_\xi\Big)_D \equiv_{\mathcal{A}} \Big(\prod_{\xi<\lambda} \mathcal{N}_\xi\Big)_D.$$

Claim 1. There exist $\alpha, \gamma < 2^\lambda$ such that for all $\xi < \lambda$,

$$\mathcal{M}_\xi \cong \mathcal{M}_\xi^{\alpha,\gamma}, \qquad \mathcal{N}_\xi \cong \mathcal{N}_\xi^{\alpha,\gamma}.$$

Proof of Claim 1. For each L_0-sentence θ, let

$$X(\theta) = \{\, \xi < \lambda \mid \mathcal{M}_\xi \models \theta \,\},$$
$$Y(\theta) = \{\, \xi < \lambda \mid \mathcal{N}_\xi \models \theta \,\},$$

and let G be the filter generated by the sets

$$X(\theta') \text{ and } Y(\theta'), \qquad \text{where } \theta' > \theta \in \mathrm{Th}_{\mathcal{A}}\Big(\Big(\prod_{\xi<\lambda} \mathcal{M}_\xi\Big)_D\Big).$$

Note that $G \subseteq D$. Moreover, G is generated by at most λ sets, so $G = G_\alpha$ for some $\alpha < 2^\lambda$. Further, for every L_0-sentence θ and every approximation θ' of θ we have

$$\{\xi < \lambda \mid \mathcal{M}_\xi \models \theta\} \in G \text{ implies } \{\xi < \lambda \mid \mathcal{N}_\xi \models \theta'\} \in G$$

and

$$\{\xi < \lambda \mid \mathcal{N}_\xi \models \theta\} \in G \text{ implies } \{\xi < \lambda \mid \mathcal{M}_\xi \models \theta'\} \in G,$$

so Claim 1 follows. □

Fix α and γ as given by Claim 1, and let

$$A^{\alpha,\gamma} = \bigcup_{i<2^\lambda} A_i^{\alpha,\gamma},$$
$$A^{\alpha,\gamma} = \bigcup_{i<2^\lambda} B_i^{\alpha,\gamma},$$
$$C = \bigcup_{i<2^\lambda} C_i.$$

By (3) and (4) (and property (II) of the function g), every bounded element of $\prod_{\xi<\lambda} M_\xi^{\alpha,\gamma}$ is of the form $A^{\alpha,\gamma}(c)$ for some $c \in C$ and every bounded element of $\prod_{\xi<\lambda} N_\xi^{\alpha,\gamma}$ is of the form $B^{\alpha,\gamma}(c)$ for some $c \in C$.

Claim 2. Every element of $(\prod_{\xi<\lambda} \mathcal{M}_\xi^{\alpha,\gamma})_D$ is of the form $(a)_D$ for some bounded element a of $\prod_{\xi<\lambda} M_\xi^{\alpha,\gamma}$.

Proof of Claim 2. The set of equivalence classes of the form $(a)_D$, where a is a bounded element of $\prod_{\xi<\lambda} M_\xi^{\alpha,\gamma}$, is dense in $(\prod_{\xi<\lambda} \mathcal{M}_\xi^{\alpha,\gamma})_D$, so we may use condition (5) to argue as in the proof of Proposition 9.21. $\square$

Claim 3. Every element of $(\prod_{\xi<\lambda} \mathcal{N}_\xi^{\alpha,\gamma})_D$ is of the form $(b)_D$ for some bounded element b of $\prod_{\xi<\lambda} N_\xi^{\alpha,\gamma}$.

Proof of Claim 3. Similar to the proof of Claim 2, but using condition (6) in place of condition (5). $\square$

Claim 4. The structure $(\prod_{\xi<\lambda} \mathcal{M}_\xi^{\alpha,\gamma})_D$ is κ^+-saturated; hence so is $(\prod_{\xi<\lambda} \mathcal{M}_\xi)_D$.

Proof of Claim 4. By Claim 2 and condition (5). $\square$

Claim 5. If $\varphi(x_1,\dots,x_n)$ is a positive bounded L_0-formula,

$$a_1,\dots,a_n \in \ell_\infty(\,\lambda,(M_\xi^{\alpha,\gamma} \mid \xi<\lambda)\,),$$
$$b_1,\dots,b_n \in \ell_\infty(\,\lambda,(N_\xi^{\alpha,\gamma} \mid \xi<\lambda)\,),$$

and $c_1,\dots,c_n \in C$ are such that

$$A^{\alpha,\gamma}(c_k) = a_k \qquad (k=1,\dots,n),$$
$$B^{\alpha,\gamma}(c_k) = b_k \qquad (k=1,\dots,n),$$

then

$$\left\{\ \xi<\lambda \ \middle|\ \mathcal{M}_\xi^{\alpha,\gamma} \models \varphi[a_1(\xi),\dots,a_n(\xi)]\ \right\} \in D$$

implies

$$\left\{\ \xi<\lambda \ \middle|\ \mathcal{N}_\xi^{\alpha,\gamma} \models \varphi'[b_1(\xi),\dots,b_n(\xi)]\ \right\} \in D \quad \text{for every } \varphi' > \varphi.$$

Proof of Claim 5. Suppose

$$\left\{\ \xi<\lambda \ \middle|\ \mathcal{M}_\xi^{\alpha,\gamma} \models \varphi[a_1(\xi),\dots,a_n(\xi)]\ \right\} \in D.$$

Let i be the least ordinal such that $g(i) = (1,\dots)$ and $a_1,\dots,a_n$ are in the range of $A_i^{\alpha,\gamma}$. Let j be such that $i = j+1$. The set $D_j \cup G_\alpha$ generates a proper filter on λ since $G_\alpha = G \subseteq D$. Hence, by condition (3)(a),

$$\left\{\ \xi<\lambda \ \middle|\ \mathcal{M}_\xi^{\alpha,\gamma} \models \varphi[a_1(\xi),\dots,a_n(\xi)]\ \right\} \in D'_j,$$

and by (3)(b),

$$\left\{ \xi < \lambda \;\middle|\; \mathcal{N}_\xi^{\alpha,\gamma} \models \varphi'[b_1(\xi), \ldots, b_n(\xi)] \right\} \in D_i \subseteq D$$

for every $\varphi' > \varphi$. $\square$

By Claims 2, 3, and 5,

$$(\prod_{\xi<\lambda} \mathcal{M}_\xi^{\alpha,\gamma})_D \cong (\prod_{\xi<\lambda} \mathcal{N}_\xi^{\alpha,\gamma})_D$$

and therefore

$$(\prod_{\xi<\lambda} \mathcal{M}_\xi)_D \cong (\prod_{\xi<\lambda} \mathcal{N}_\xi)_D.$$

by Claim 1. $\square$

10.9 Remark. An important case of Theorems 10.7 and 10.8 and their consequences is when $\kappa = \aleph_0$. In this case card$(L) \leq 2^{\aleph_0}$ and we are considering normed space L-structures $\mathcal{M}$ such that every sort of $\mathcal{M}$ is separable. In this situation, the ultrafilter D in Theorems 10.7 and 10.8 lives on an index set of cardinality equal to the continuum $c = 2^{\aleph_0}$. In the ultrapowers and ultraproducts considered in these results, each sort has density character at most 2^c.

10.10 Remark. For Banach spaces and Banach lattices (but not for normed space structures in general) Theorem 10.7 was first proved in [Hen75] (see especially Theorem 4) and was further treated in [Hen76] (see Theorem 1.13). See also [HM83a] where several characterizations of $\equiv_{\mathcal{A}}$ for Banach spaces are discussed. Those proofs applied the Keisler-Shelah Theorem together with Lyndon's Theorem from ordinary model theory to give an indirect proof of the Banach space result, rather than constructing a direct proof as is done here. Stern [Ste78] (see also [Ste75]) applied the Keisler-Shelah Theorem in a similar way to prove that if X is a Banach space and U, V are ultrafilters, then there exists an ultrafilter D such that $((X)_U)_D$ and $((X)_V)_D$ are isometrically isomorphic. This is, of course, a special case of Theorem 10.7. Shelah's full result for ultraproducts, and its analogue Theorem 10.8 for normed spaced structures, are closely connected to the Isomorphism Properties [Hen74] [HM83b] used in nonstandard analysis. Another characterization of $\equiv_{\mathcal{A}}$ can be given in terms of Ehrenfeucht-Fraissé Games. Discussions of this topic in the restricted setting of pure Banach spaces can be found in [HM83a], [HM83b], and [HH86]. For a general signature L, the results and proofs are essentially the same as the ones given in those papers.

11

Alternative Formulations of the Theory

It is possible to handle the role of the scalar field $\mathbb{R}$ in a number of different ways, without changing the overall expressive power of this logic. For example, we might have included constant symbols in every signature for all the real numbers, as opposed to only the rationals. (We chose to do the latter in order to avoid artificially increasing the cardinality of signatures, since this has an effect on such results as the Downward Löwenheim-Skolem Theorem (Proposition 9.13)).

Using the isomorphism theorem for ultrapowers (Theorem 10.7) we can show that adding all the reals as constants does not change approximate elementary equivalence of normed space structures, although it does increase the set of positive bounded formulas. (The Perturbation Lemma (Proposition 5.15) can also be used to prove this.)

11.1 Proposition. *Let L be a signature and let $C = \{c_r \mid r \in \mathbb{R}\}$ be a new set of constant symbols in 1-1 correspondence with $\mathbb{R}$; for each $r \in \mathbb{R}$ the constant symbol c_r has the sort $s_{\mathbb{R}}$. For each normed space L-structure $\mathcal{M}$ let $\mathcal{M}'$ be the $L(C)$-structure that expands $\mathcal{M}$ and interprets c_r as r for each $r \in \mathbb{R}$. Then, for any L-structures $\mathcal{M}$ and $\mathcal{N}$,*

$$\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N} \;\; \textit{if and only if} \;\; \mathcal{M}' \equiv_{\mathcal{A}} \mathcal{N}'.$$

and

$$\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N} \;\; \textit{if and only if} \;\; \mathcal{M}' \preceq_{\mathcal{A}} \mathcal{N}'.$$

Proof. Since $\mathcal{M}$ is a reduct of $\mathcal{M}'$ for each normed space L-structure $\mathcal{M}$, the right to left directions of these equivalences are immediate. We prove the left to right implications using the isomorphism theorem for ultrapowers. Let $\mathcal{M}$, $\mathcal{N}$ be L-structures such that $\mathcal{M} \equiv_{\mathcal{A}} \mathcal{N}$. Take κ to be a cardinal number greater than the density character of each sort of $\mathcal{M}$ and of $\mathcal{N}$, and such that $\operatorname{card}(L) \leq \kappa$. By Theorem 10.7 there is an ultrafilter D on $\lambda = 2^\kappa$ such that $(\mathcal{M})_D \cong (\mathcal{N})_D$. It follows that $((\mathcal{M})_D)' \cong ((\mathcal{N})_D)'$. One easily sees that $(\mathcal{M}')_D \cong ((\mathcal{M})_D)'$ and the same for $\mathcal{N}$. Therefore $(\mathcal{M}')_D \cong (\mathcal{N}')_D$. By Corollary 9.4 it follows that $\mathcal{M}' \equiv_{\mathcal{A}} \mathcal{N}'$.

The second implication follows from the first when it is applied to an expansion by constants of $\mathcal{M}$ in which every element of $\mathcal{M}$ is given a name. $\square$

Note that a similar argument shows that if we add every continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ to a normed space structure $\mathcal{M}$, then the positive bounded theory of the new structure is determined by $\mathrm{Th}_{\mathcal{A}}(\mathcal{M})$; we would make this precise using the same language as in Proposition 11.1. This can sometimes be useful when characterizing properties of normed space structures explicitly by positive bounded sentences.

Similar considerations show that it does not change the expressive power of this logic to completely omit the real sort, as long as this is done carefully. To illustrate what is involved here, we consider only the example of Banach spaces with no additional structure. Let L_0 be the signature for such structures. (See Chapter 7.) We call an L_0-formula *restricted* if it does not contain any variables or constant symbols whose sort is $s_{\mathbb{R}}$, and if the only function symbols it contains are the norm and the 0 and + of the normed space sort. Up to equivalence in every normed space L_0-structure, the restricted atomic formulas are $\|k_1x_1 + \cdots + k_nx_n\| \leq r$ and $r \leq \|k_1x_1 + \cdots + k_nx_n\|$, where $k_1, \ldots, k_n$ are integers, r is a rational number, and $x_1, \ldots, x_n$ are variables of the normed space sort. Restricted formulas in general are constructed from restricted atomic formulas using $\wedge, \vee$ and bounded quantifiers $\exists_r x, \forall_r x$ where r is a positive rational number and x is a variable of the normed space sort. Note that the weak negation and every approximation of a restricted L_0-formula are restricted.

The following result shows that the overall expressive power of restricted formulas is the same as that of the full class of positive bounded L_0-formulas.

11.2 Proposition. *Let X, Y be normed spaces, considered as normed space L_0-structures.*

(i) *Suppose that for every restricted L_0-sentence σ,*

$$X \models_{\mathcal{A}} \sigma \quad \textit{if and only if} \quad Y \models_{\mathcal{A}} \sigma.$$

Then $X \equiv_{\mathcal{A}} Y$*.*

(ii) *Assume* $X \subseteq Y$*. Suppose that for every* $a_1, \ldots, a_n \in X$ *and every restricted* L_0*-formula* $\varphi(x_1, \ldots, x_n)$*,*

$$X \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad Y \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n].$$

Then $X \preceq_{\mathcal{A}} Y$*.*

Proof. We prove (i); (ii) is proved similarly. Assume that X, Y satisfy the hypotheses of (i). We indicate how the proof of Theorem 10.7 can be modified to prove that there is an ultrafilter D such that $(X)_D \cong (Y)_D$. From this it follows that $X \equiv_{\mathcal{A}} Y$, as desired, using Corollary 9.4.

Take κ to bound the cardinalities of X and Y, and then construct an ultrafilter D on $\lambda = 2^\kappa$ as in the proof of Theorem 10.7. The modification of the proof consists of replacing every reference to *positive bounded formula* in the inductive construction by *restricted positive bounded formula.* At the end of the construction one has a mapping $T\colon \ell_\infty(\lambda, X) \to \ell_\infty(\lambda, Y)$ such that for every restricted positive bounded L_0-formula $\varphi(x_1, \ldots, x_n)$, every approximation φ' of φ, and every $a_1, \ldots, a_n \in \ell_\infty(\lambda, X)$,

$$\{\xi < \lambda \mid X \models \varphi[a_1(\xi), \ldots, a_n(\xi)]\} \in D$$

implies

$$\{\xi < \lambda \mid Y \models \varphi'[T(a_1)(\xi), \ldots, T(a_n)(\xi)]\} \in D.$$

Using the expressive power of restricted atomic formulas, it follows immediately that T induces an isomorphism between $(X)_D$ and $(Y)_D$ as L_0-structures. □

The previous argument can be applied in a much more general setting, and this is only a limited example of how one can vary the handing of the scalar sort without changing the overall expressive power of the logic.

We note that restricted L_0-formulas are (up to equivalence in all normed space L_0-structures) among the positive bounded formulas for the normed space setting that were considered in earlier papers such as [Hen76] [HM83b] [HH86]. Moreover, the concept of approximate satisfaction considered in those papers is easily seen to be equivalent to the one used here. From Proposition 11.2 it follows that the equivalence relation $\equiv_{\mathcal{A}}$ for the signature L_0 that is treated in this paper is the same as the one considered in those earlier papers (which only considered Banach spaces without additional structure).

12

Homogeneous Structures

12.1 Definition. Let $\mathcal{M}$ be a normed space L-structure and let κ be an infinite cardinal. We say that $\mathcal{M}$ is *strongly κ-homogeneous* if the following condition holds: whenever $L(C)$ is an extension of L by constants with $\operatorname{card}(C) < \kappa$ and $\mathbf{A}, \mathbf{B}$ are interpretations of C in $\mathcal{M}$ such that

$$(\mathcal{M}, \mathbf{A}) \equiv_{\mathcal{A}} (\mathcal{M}, \mathbf{B})$$

one has

$$(\mathcal{M}, \mathbf{A}) \cong (\mathcal{M}, \mathbf{B}).$$

Note that an isomorphism from $(\mathcal{M}, \mathbf{A})$ onto $(\mathcal{M}, \mathbf{B})$ is an *automorphism* of $\mathcal{M}$ that takes $\mathbf{A}(c)$ to $\mathbf{B}(c)$ for each $c \in C$.

12.2 Proposition. *Let κ be an infinite cardinal number and let $\lambda = 2^\kappa$. Then the ultrafilter D on λ given by Theorem 10.8 satisfies the following property: Whenever we are given*

- *a signature L with $\operatorname{card}(L) \leq \lambda$, and*
- *a family of normed space L-structures $(\mathcal{M}_\xi \mid \xi < \lambda)$ such that the density character of each sort of $\mathcal{M}_\xi$ is at most κ for every $\xi < \lambda$,*

then the ultraproduct $(\prod_{\xi<\lambda} \mathcal{M}_\xi)_D$ is always κ^+-saturated and strongly λ^+-homogeneous.

Before proving Proposition 12.2, we note the following important corollary:

12.3 Corollary. *Let κ be an infinite cardinal number and let $\lambda = 2^\kappa$. Then there exists an ultrafilter D on λ satisfying the following property: whenever $\mathcal{M}$ is an L-structure such that $\operatorname{card}(L) \leq \lambda$ and the density character of each sort of $\mathcal{M}$ is at most κ, the ultrapower $(\mathcal{M})_D$ is κ^+-saturated and strongly λ^+-homogeneous.*

Proof. By Propositions 6.7 and 12.2. □

Proof of Proposition 12.2. The κ^+-saturation of $(\prod_{\xi<\lambda}\mathcal{M}_\xi)_D$ is given by the last part of Theorem 10.8. We now prove that $(\prod_{\xi<\lambda}\mathcal{M}_\xi)_D$ is strongly λ^+-homogeneous. To this end, fix an extension by constants $L(C)$ of L with $\operatorname{card}(C)\leq\lambda$. Note that $\operatorname{card}(L(C))\leq\lambda$.

Suppose that $\mathbf{A}$ is any interpretation of C in $(\prod_{\xi<\lambda}\mathcal{M}_\xi)_D$. The key observation for this proof is that there exist interpretations $\mathbf{A}_\xi$ of C in $\mathcal{M}_\xi$ for each $\xi<\lambda$ such that $\mathbf{A}(c) = (\,(\mathbf{A}_\xi(c))_{\xi<\lambda}\,)_D$ for every $c\in C$ and hence that

$$\Big(\,\big(\prod_{\xi<\lambda}\mathcal{M}_\xi\big)_D, \mathbf{A}\Big) = \Big(\prod_{\xi<\lambda}(\mathcal{M}_\xi,\mathbf{A}_\xi)\Big)_D.$$

Suppose that $\mathbf{A},\mathbf{B}$ are interpretations of C in $(\prod_{\xi<\lambda}\mathcal{M}_\xi)_D$ such that

$$\Big(\,\big(\prod_{\xi<\lambda}\mathcal{M}_\xi\big)_D, \mathbf{A}\Big) \equiv_{\mathcal{A}} \Big(\,\big(\prod_{\xi<\lambda}\mathcal{M}_\xi\big)_D, \mathbf{B}\Big).$$

Apply the previous remark to $\mathbf{A}$ and $\mathbf{B}$, obtaining $\mathbf{A}_\xi$ and $\mathbf{B}_\xi$ for each $\xi<\lambda$ such that

$$\Big(\,\big(\prod_{\xi<\lambda}\mathcal{M}_\xi\big)_D, \mathbf{A}\Big) = \Big(\prod_{\xi<\lambda}(\mathcal{M}_\xi,\mathbf{A}_\xi)\Big)_D$$

and

$$\Big(\,\big(\prod_{\xi<\lambda}\mathcal{M}_\xi\big)_D, \mathbf{B}\Big) = \Big(\prod_{\xi<\lambda}(\mathcal{M}_\xi,\mathbf{B}_\xi)\Big)_D.$$

From this it follows that

$$\Big(\prod_{\xi<\lambda}(\mathcal{M}_\xi,\mathbf{A}_\xi)\Big)_D \equiv_{\mathcal{A}} \Big(\prod_{\xi<\lambda}(\mathcal{M}_\xi,\mathbf{B}_\xi)\Big)_D.$$

Therefore,

$$\Big(\,\big(\prod_{\xi<\lambda}\mathcal{M}_\xi\big)_D, \mathbf{A}\Big) \cong \Big(\,\big(\prod_{\xi<\lambda}\mathcal{M}_\xi\big)_D, \mathbf{B}\Big)$$

by Theorem 10.8 and the equations above. □

12.4 Remark. A technically easier proof that κ-saturated, strongly κ-homogeneous approximate elementary extensions exist can be given using the existence of highly saturated extensions and a union-of-chains argument, as we now outline. We may assume κ is regular, increasing κ if necessary. Given any normed space L-structure $\mathcal{M}$, we construct an approximate elementary chain $(\mathcal{M}_\alpha \mid \alpha<\kappa)$ whose union has the desired properties. Let $\mathcal{M}_0=\mathcal{M}$; for each $\alpha<\kappa$, let $\mathcal{M}_{\alpha+1}$ be an approximate elementary extension of $\mathcal{M}_\alpha$ that is τ_α-saturated, where τ_α is a cardinal bigger than $\mathrm{card}(L)$ and bigger than the cardinality of each sort of $\mathcal{M}_\alpha$; take unions at limit ordinals. Let $\mathcal{N}$ be the union of $(\mathcal{M}_\alpha \mid \alpha < \kappa)$. By Corollary 9.12, $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$. An argument such as in the proof of Proposition 9.25 shows that $\mathcal{N}$ is κ-saturated. The fact that $\mathcal{N}$ is strongly κ-homogeneous follows from an inductive argument whose successor steps are based on the following easily proved fact:

Suppose $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$ and $\mathcal{N}$ is τ-saturated, where τ is a cardinal bigger than $\mathrm{card}(L)$ and bigger than the cardinality of each sort of $\mathcal{M}$. Let C be a set of new constants with $\mathrm{card}(C) < \tau$. Suppose $\mathbf{A}, \mathbf{B}$ are interpretations of C in $\mathcal{M}$ such that $(\mathcal{M}, \mathbf{A}) \equiv_{\mathcal{A}} (\mathcal{N}, \mathbf{B})$. Then there exists an approximate elementary embedding $T \colon \mathcal{M} \to \mathcal{N}$ such that for every $c \in C$, $T(\mathbf{A}(c)) = \mathbf{B}(c)$.

12.5 Remark. S. Buechler and O. Lessmann have developed an approach to geometric stability theory in the absence of full model theoretic compactness [BL] . We briefly indicate how this can be applied to normed space structures. Let L be any signature and let $\mathcal{M}_0$ be a normed space L-structure. Let λ be a cardinal number $\geq \aleph_1$. Applying Corollary 12.3 to $\mathcal{M}_0$ and a sufficiently large κ, let $\mathcal{M}$ be a Banach space ultrapower of $\mathcal{M}_0$ such that $\mathcal{M}$ is λ-saturated and strongly λ-homogeneous. We regard $\mathcal{M}$ as a logical structure $(M, \mathcal{R})$ in the sense of [BL, Definition 1.1] by taking $\mathcal{R}$ to be the collection of all finitary relations that are definable in $\mathcal{M}$ by positive bounded L-formulas (without parameters except for the ones named by constant symbols of L). (Recall that since $\mathcal{M}$ is at least $\aleph_1$-saturated, $\models_{\mathcal{A}}$ is equivalent to $\models$ in $\mathcal{M}$ for positive bounded formulas; see Proposition 9.20.) It is easy to verify that $(M, \mathcal{R})$ satisfies the concept of strong λ-homogeneity used by Buechler and Lessmann (see [BL, Definition 1.3]) and therefore that their methods apply to this structure. Stability and simplicity of the original structure $\mathcal{M}_0$ can be studied by applying the definitions and methods from [BL] to $\mathcal{M}$. (It is not difficult to show that this is independent of $\mathcal{M}$, as long as λ is sufficiently large.)

In the mid-1990s, the second author developed a concept of stability for normed space structures in the context of the model theory developed here; see [Iov94] [Iov96] [Iov98] [Iov99a] [Iov99b]. Using what is presented in those papers (especially in [Iov99b, Section 4]) and in [BL], and adapting certain other arguments in the literature to the setting of positive bounded formulas and normed space structures, it can be shown that these two notions of stability for normed space structures are equivalent.

A number of interesting normed space structures are stable in this sense. For example, a type-counting argument based on simple features of Hilbert space geometry can be used to show that if $\mathcal{M}$ consists of a Hilbert space H together with a family $(T_i \mid i \in I)$ of bounded linear operators $T_i \colon H \to H$, then $\mathcal{M}$ is stable. Another example of a stable normed space structure is the Banach lattice $L_p(\mu)$, where $1 \leq p < \infty$ and μ is any measure.

An explanation of the equivalence of these two notions of stability for normed space structures and a treatment of these examples will be given elsewhere.

13

More Model Theory

Quantifier Complexity of Formulas

For every positive bounded L-formula $\varphi(x_1, \ldots, x_n)$ there exists a positive bounded L-formula $\psi(x_1, \ldots, x_n)$ such that ψ is of the form

$$\mathsf{Q}_1 y_1 \mathsf{Q}_2 y_2 \ldots \mathsf{Q}_k y_k \, \theta(x_1, \ldots, x_n, y_1, \ldots, y_k)$$

where

- Q_i is a bounded quantifier of the form $\exists_{r_i}$ or $\forall_{r_i}$, for $i = 1, \ldots, k$;
- the positive bounded formula $\theta(x_1, \ldots, x_n, y_1, \ldots, y_k)$ is quantifier-free;
- $\psi(x_1, \ldots, x_n)$ is equivalent to $\varphi(x_1, \ldots, x_n)$ in the sense that for every normed space L-structure $\mathcal{M}$ and elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}$,

$$\mathcal{M} \models \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathcal{M} \models \psi[a_1, \ldots, a_n].$$

The formula ψ is said to be in *prenex form*. We say that ψ is *existential* if all its quantifiers are existential. Similarly, we say that ψ is *universal* if all its quantifiers are universal.

Note that any approximation of a prenex form positive bounded formula will also be in prenex form, with exactly the same sequence of leading quantifier types ($\forall$ versus $\exists$).

Note also that the usual procedure for putting a given formula into prenex form, when applied to a positive bounded formula, commutes nicely with the process of forming approximations. If φ is a positive bounded formula and φ' is an approximation of φ, then the logic structure of φ' is identical to that of φ. Suppose ψ is in prenex form and is equivalent to φ, as above; let ψ' be obtained from φ' by applying the same sequence of prenex rules that was used to obtain ψ from φ. Then

ψ' is equivalent to φ' and ψ' is an approximation of ψ. This observation allows us to investigate the approximate satisfaction of the positive bounded formula φ by replacing φ with its equivalent prenex form ψ.

Axiomatizing Classes of Substructures

If φ is a universal sentence and $\mathcal{M}$ satisfies φ, then every substructure of $\mathcal{M}$ satisfies φ too. The following result shows that this is a characteristic way of describing uniform classes of L-structures that are closed under isomorphism and substructures:

13.1 Proposition. *Suppose that $\mathfrak{D}$ is a uniform class of normed space L-structures and let Γ be the set of universal L-sentences that are satisfied by every structure in $\mathfrak{D}$. Let $\mathfrak{C}$ be the class of L-structures that are isomorphic to a substructure of an ultraproduct of members of $\mathfrak{D}$. Then $\mathfrak{C}$ consists of the L-structures that satisfy Γ.*

Proof. Every structure in $\mathfrak{C}$ satisfies Γ, so we just need to prove the converse. Fix an L-structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$. Let C be a set of new constants and $\mathbf{A}$ an interpretation of C in $\mathcal{M}$ such that every element of $\mathcal{M}$ is of the form $\mathbf{A}(c)$ for some $c \in C$. Let Δ be the set of all quantifier-free positive bounded sentences in $L(C)$ that are true in $(\mathcal{M}, \mathbf{A})$. Let Σ be the set of all finite subsets of Δ^+.

Consider $\xi = \{\sigma_1, \ldots, \sigma_n\} \in \Sigma$. There exist quantifier-free positive bounded L- formulas $\psi_j(x_1, \ldots, x_m)$ for $j = 1, \ldots, n$ and $c_1, \ldots, c_m \in C$ such that for each $j = 1, \ldots, n$, the formula σ_j is equal to $\psi_j(c_1, \ldots, c_m)$. Note that x_i and c_i must have the same sort for $i = 1, \ldots, m$. Moreover, for each $j = 1, \ldots, n$ there exists a positive bounded L-formula φ_j with $\varphi_j < \psi_j$ and $\varphi_j(c_1, \ldots, c_m) \in \Delta$. For each $i = 1, \ldots, m$ let r_i be a rational number with $\|\mathbf{A}(c_i)\| \leq r_i$. Without loss of generality we may assume that each φ_j contains the formula $\|x_i\| \leq r_i$ as a conjunct. Let $s_i > r_i$ be the rational number for which each approximation ψ_j contains $\|x_i\| \leq s_i$ as a conjunct in the corresponding place.

We claim that there exist an L-structure $\mathcal{N}_\xi \in \mathfrak{D}$ and an interpretation $\mathbf{A}_\xi$ of C in $\mathcal{N}_\xi$ such that the $L(C)$-structure $(\mathcal{N}_\xi, \mathbf{A}_\xi)$ satisfies every element of ξ. Otherwise we would have

$$\forall_{s_1} x_1 \ldots \forall_{s_m} x_m \bigvee_{i=1}^{n} \operatorname{neg}(\psi_i) \in \Gamma,$$

and hence this universal sentence would be true in $\mathcal{M}$. However, this is impossible, since by our choice of formulas we have

$$\mathcal{M} \not\models \bigvee_{i=1}^{n} \operatorname{neg}(\psi_i)[\mathbf{A}(c_1), \dots, \mathbf{A}(c_m)].$$

Let D be an ultrafilter on Σ that contains the set

$$\{\xi \in \Sigma \mid \psi \in \xi\}$$

whenever $\psi \in \Delta^+$. Then the D-ultraproduct of $(\mathcal{N}_\xi \mid \xi \in \Lambda)$ contains a substructure isomorphic to $\mathcal{M}$. Therefore $\mathcal{M}$ is in $\mathfrak{C}$. □

13.2 Definition. Suppose that $\mathfrak{C}$ is a uniform class of normed space L-structures and there exists a set Γ of universal positive bounded L-sentences such that $\mathfrak{C}$ is exactly the class of L-structures that satisfy Γ. In this situation we will say that $\mathfrak{C}$ *is axiomatizible by universal L-sentences.*

13.3 Corollary. *Let $\mathfrak{C}$ be a uniform class of L-structures. Then $\mathfrak{C}$ is axiomatizable by universal L-sentences if and only if it is closed under isomorphism, under ultraproducts, and under taking substructures.*

Proof. ($\Leftarrow$) Apply Proposition 13.1 with $\mathfrak{D} = \mathfrak{C}$. □

13.4 Examples. The literature contains a number of important examples of classes of normed space structures which can be shown to have axiomatizations by universal positive bounded sentences. In some cases the conditions in Corollary 13.3 are shown to be true of the class in question, and so one has a "soft" proof that the axiomatization exists. In other cases explicit conditions for membership in the class are given that can be expressed using universal (positive bounded) sentences (sometimes easily, sometimes with effort).

- Fix p in the interval $1 \leq p < \infty$. [BDCK66] [DCK72] show that the class of Banach lattices isomorphic to $L_p(\mu)$ for some measure μ is axiomatizable by universal sentences in the signature of Banach lattices. (Also see the discussion of abstract L_p-spaces in [MN91] and [Lac74].)
- [Hen76] [Ste76] show that the class of Banach lattices (with a designated element) isomorphic to $C(K)$ (with constant function 1) for some compact Hausdorff space K is axiomatizable by universal sentences in the signature of Banach lattices with a designated element. (Also see the discussion of abstract M-spaces in [MN91] and [Lac74].)

- Fix p in the interval $1 \leq p < \infty$. [BDCK65] [BDCK66] [Kri67] show that the class of Banach spaces embedable into $L_p(\mu)$ for some measure μ is axiomatizable by universal sentences in the signature of pure Banach spaces.
- [DCK70] considers a class $\mathfrak{D}$ of Banach spaces closed under ultraproducts, and shows that the class of Banach spaces λ-embedable in a member of $\mathfrak{D}$ can be axiomatized by conditions expressible by universal sentences in the signature of pure Banach spaces. See also [DC72a] (especially Theorem 1) and [DCK72].
- [DC72b] gives a characterization due to Krivine of embedability in classes of "ψ-spaces" that are closed under ultraproduct and "ψ-sums", using conditions that can be expressed by universal sentences in the expansion of the signature of pure Banach spaces obtained by adding a function symbol for the function ψ.

Axiomatizability of Classes of Structures

13.5 Definition. Suppose that $\mathfrak{C}, \mathfrak{D}$ are classes of normed space L-structures such that $\mathfrak{C} \subseteq \mathfrak{D}$. We say that $\mathfrak{C}$ is *axiomatizable in* $\mathfrak{D}$ *by positive bounded sentences* if there exists a set Γ of positive bounded L-sentences such that the structures in $\mathfrak{C}$ are exactly those normed space L-structures in $\mathfrak{D}$ which approximately satisfy Γ. When this condition holds for Γ, we say that Γ is a set of *positive bounded axioms* for $\mathfrak{C}$ in $\mathfrak{D}$.

In this section we characterize axiomatizability by positive bounded sentences in terms of ultraproducts. The ideas are patterned after a well known characterization of axiomatizability in first order logic due to Keisler [Kei61]. (See Corollary 6.1.16 in [CK90].)

13.6 Proposition. *Suppose that* $\mathfrak{C}, \mathfrak{D}$ *are classes of normed space* L*-structures with* $\mathfrak{C} \subseteq \mathfrak{D}$ *such that* $\mathfrak{D}$ *uniform and is closed under isomorphisms and ultraproducts. The following conditions are equivalent:*

(i) $\mathfrak{C}$ *is axiomatizable in* $\mathfrak{D}$ *by positive bounded sentences;*

(ii) $\mathfrak{C}$ *is closed under isomorphisms and ultraproducts, and* $\mathfrak{D} \setminus \mathfrak{C}$ *is closed under ultrapowers.*

Proof. (i)$\Rightarrow$(ii) follows from Propositions 5.14 and 9.3.

To prove (ii)$\Rightarrow$(i), we let Γ be the set of positive bounded L-sentences that are satisfied by every structure in $\mathfrak{C}$. We claim that Γ is a set of positive bounded axioms for $\mathfrak{C}$ in $\mathfrak{D}$. To prove this, suppose that $\mathcal{M}$ is a structure in $\mathfrak{D}$ such that $\mathcal{M} \models_{\mathcal{A}} \Gamma$. Let

$$\Sigma = \left\{ \{\psi_1, \ldots, \psi_n\} \mid \psi_1, \ldots, \psi_n \in (\mathrm{Th}_{\mathcal{A}}(\mathcal{M}))^+,\ n \in \mathbb{N} \right\}.$$

If $\xi = \{\psi_1, \ldots, \psi_n\} \in \Sigma$, then there exists a structure in $\mathfrak{C}$ which satisfies every element of ξ; otherwise we would have $\bigvee_{i=1}^n \mathrm{neg}(\psi_i) \in \Gamma$, so $\mathcal{M} \models_{\mathcal{A}} \bigvee_{i=1}^n \mathrm{neg}(\psi_i)$; but this is impossible because if $\varphi_1, \ldots, \varphi_n \in \mathrm{Th}_{\mathcal{A}}(\mathcal{M})$ are such that $\varphi_i < \psi_i$ for $i = 1, \ldots, n$, we have $\mathcal{M} \models_{\mathcal{A}} \bigwedge_{i=1}^n \varphi_i$, which is a contradiction.

For each $\xi = \{\psi_1, \ldots, \psi_n\} \in \Sigma$ let $\mathcal{N}_\xi$ be a structure in $\mathfrak{C}$ which satisfies every sentence in ξ. Let D be an ultrafilter on Σ that contains the set

$$\{\xi \in \Sigma \mid \psi \in \xi\}$$

for each $\psi \in \mathrm{Th}_{\mathcal{A}}(\mathcal{M}))^+$. Let $\mathcal{K} = (\prod_{\xi \in \Lambda} \mathcal{N}_\xi)_D$. Then $\mathcal{K} \models_{\mathcal{A}} \mathrm{Th}_{\mathcal{A}}(\mathcal{M})$, so we have $\mathcal{K} \equiv_{\mathcal{A}} \mathcal{M}$. Since $\mathfrak{C}$ is closed under ultraproducts, $\mathcal{K}$ is in $\mathfrak{C}$. By Theorem 10.7, there exists an ultrafilter D such that $(\mathcal{K})_D$ and $(\mathcal{M})_D$ are isomorphic. Now $(\mathcal{K})_D$ is in $\mathfrak{C}$, so $(\mathcal{M})_D$ is in $\mathfrak{C}$, since $\mathfrak{C}$ is closed under isomorphisms. Therefore, since $\mathfrak{D} \setminus \mathfrak{C}$ is closed under ultrapowers, $\mathcal{M}$ is also in $\mathfrak{C}$. □

13.7 Remark. The proof of Proposition 13.6 contains the following useful elementary result: let $\mathfrak{C}$ be a uniform class of L-structures and let Γ be the set of all positive bounded sentences σ such that $\mathcal{M} \models_{\mathcal{A}} \sigma$ holds for all $\mathcal{M} \in \mathfrak{C}$. Then, every approximate model of Γ is approximately elementarily equivalent to some ultraproduct of structures from $\mathfrak{C}$.

13.8 Examples. The literature contains a number of important examples of classes of normed space structures with axiomatizations by positive bounded sentences.

- [HM74a] [Ste76] [Ste78] contain results from which one can show that the class of Banach spaces isometric to $L_p(\mu)$ for some measure μ can be axiomatized by positive bounded sentences in the signature of pure Banach spaces (p is fixed, in the interval $1 \le p < \infty$). The key idea is that these spaces are the same as the spaces that are $\mathcal{L}_{p,\lambda}$-spaces for all $\lambda > 1$. A full analysis of the complete positive bounded theories of Banach spaces in these classes is discussed in [Hen76]; the added information needed to axiomatize a complete theory is the "number" of atoms of the measure μ (this "number" is either an integer ≥ 0 or ∞). (For a discussion of $\mathcal{L}_{p,\lambda}$-spaces, needed in this paragraph and the next, see Chapter II.5 in [LT73].)

- The papers cited in the previous item also cover the case $p = \infty$. The axiomatizable class here consists of all Banach spaces whose dual space is isometric to an $L_1(\mu)$-space; these are the same as the spaces that are $\mathcal{L}_{\infty,\lambda}$-spaces for all $\lambda > 1$. The classification of these Banach spaces under $\equiv_{\mathcal{A}}$ is more or less completely open.
- The results in [Hei81] show that the class of Banach spaces isometric to $C(K)$ for some compact Hausdorff space K can be axiomatized by positive bounded sentences in the signature of pure Banach spaces. The key result in this paper is that the class of $C(K)$ Banach spaces is closed under ultraroots. An application of Proposition 13.6 yields that the required set of positive bounded sentences exists; no explicit set of axioms is known. Certain properties of K are known to be axiomatizable in this sense, such as connectedness and total disconnectedness (see [Hen76]), but a complete classification of $C(K)$ Banach spaces under $\equiv_{\mathcal{A}}$ is far from being known. Some partial results are given in [HHM83] [HHM86] [HHM87].

13.9 Remark. There are a number of uniform classes of normed space structures that have been shown to be closed under ultraproducts, but for which the corresponding closure under "ultraroots" is either false or not known to be true. These provide open problems as to the (optimal) applicability of Proposition 13.6. Among the many papers in which the structure of ultraproducts of spaces of interest in functional analysis is studied, we mention the following: [DCK72] [Ray97] for Orlicz and Köthe function spaces; [LR84a] [LR84b] [HLR85] [Ray86] [HLR91]for vector valued L_p-spaces, in particular generalized $L_p(L_q)$-spaces; [Ray91] [Ray98b] for interpolation spaces; and [Gro84] [Ray98a] [Ray02] for noncommutative L_p-spaces.

Quantifier Elimination

13.10 Definition. Let Γ be a positive bounded theory in a signature L and let $\varphi(x_1, \ldots, x_n)$ be a positive bounded L-formula. We say that φ is *approximable in* Γ *by quantifier-free formulas* if the following condition holds: for every approximation φ' of φ and every positive real number r there exist a quantifier-free positive bounded L-formula $\theta(x_1, \ldots, x_n)$ and an approximation θ' of θ such that whenever $\mathcal{M}$ is an approximate model of Γ and $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$ with $\|a_j\| \leq$

r for $j = 1, \ldots, n$,

$$\mathcal{M} \models \varphi[a_1, \ldots, a_n] \quad \text{implies} \quad \mathcal{M} \models \theta[a_1, \ldots, a_n],$$
$$\mathcal{M} \models \theta'[a_1, \ldots, a_n] \quad \text{implies} \quad \mathcal{M} \models \varphi'[a_1, \ldots, a_n].$$

13.11 Remark. In Definition 13.10, one may replace $\models$ by $\models_{\mathcal{A}}$ without changing the concept being defined.

13.12 Proposition. *Let Γ be a positive bounded theory in a signature L such that the class of approximate models of Γ is uniform, and let $\varphi(x_1, \ldots, x_n)$ be a positive bounded L-formula, where x_j is a variable of sort s_j, for $j = 1, \ldots, n$. The following conditions are equivalent.*

(i) *φ is approximable in Γ by quantifier-free formulas;*

(ii) *Whenever we are given*

- *two approximate models $\mathcal{M}$ and $\mathcal{N}$ of Γ,*
- *substructures $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{N}_0 \subseteq \mathcal{N}$,*
- *an isomorphism $(T^{(s)} \mid s \in S)$ of $\mathcal{M}_0$ onto $\mathcal{N}_0$, and*
- *elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}_0$,*

we have

$$\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \quad \textit{implies} \quad \mathcal{N} \models_{\mathcal{A}} \varphi[T^{(s_1)}(a_1), \ldots, T^{(s_1)}(a_n)].$$

Moreover, for the implication (ii)$\Rightarrow$(i) *it suffices to assume* (ii) *only when $\mathcal{M}_0$ and $\mathcal{N}_0$ are finitely generated substructures.*

Proof. We omit the straightforward proof of (i)$\Rightarrow$(ii). To prove (ii)$\Rightarrow$(i), fix an approximation φ' of φ and $r > 0$ in order to find a pair θ, θ' satisfying the implications in Definition 13.10.

Let $\Sigma(x_1, \ldots, x_n)$ be the set of all quantifier-free L-formulas of the form $\operatorname{neg}(\sigma)(x_1, \ldots, x_n)$, where σ satisfies the following condition: there exists an approximation τ of σ such that whenever $\mathcal{N}$ is an approximate model of Γ and $b_1, \ldots, b_n$ are elements of suitable sorts of $\mathcal{N}$ with $\|b_j\| \leq r$ for $j = 1, \ldots, n$,

$$\mathcal{N} \models \tau[b_1, \ldots, b_n] \quad \text{implies} \quad \mathcal{N} \models \varphi'[b_1, \ldots, b_n].$$

We claim that the set

$$\{\bigwedge_{j=1}^{n} \|x_j\| \leq r\} \cup \{\, \varphi(x_1, \ldots, x_n) \,\} \cup \Sigma(x_1, \ldots, x_n)$$

is not satisfiable in any model of Γ. To show this, suppose that $\mathcal{M}$ is an approximate model of Γ and $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$ with $\|a_j\| \leq r$ for $j = 1, \dots, n$ such that $\mathcal{M} \models \varphi[a_1, \dots, a_n]$ and $\mathcal{M} \models \Sigma[a_1, \dots, a_n]$. Let $p(x_1, \dots, x_n)$ be the set of all quantifier-free L-formulas $\sigma(x_1, \dots, x_n)$ such that $\mathcal{M} \models \sigma[a_1, \dots, a_n]$, and suppose that $\mathcal{N}$ is an approximate model of Γ and $b_1, \dots, b_n$ are elements of suitable sorts of $\mathcal{N}$ with $\|b_j\| \leq r$ for $j = 1, \dots, n$ such that $\mathcal{N} \models p[b_1, \dots, b_n]$. Then the map $a_j \mapsto b_j$ determines an isomorphism between the substructure of $\mathcal{M}$ generated by $\{a_1, \dots, a_n\}$ and the substructure of $\mathcal{N}$ generated by $\{b_1, \dots, b_n\}$. Therefore, by (ii), we have $\mathcal{N} \models \varphi'[b_1, \dots, b_n]$.

Thus, by the Compactness Theorem (Proposition 9.5), there exist $\sigma \in p$ and an approximation τ of σ such that if $\mathcal{N}$ is an approximate model of Γ and $b_1, \dots, b_n$ are elements of suitable sorts of $\mathcal{N}$ with $\|b_j\| \leq r$ for $j = 1, \dots, n$, we have

$$\mathcal{N} \models \tau[b_1, \dots, b_n] \quad \text{implies} \quad \mathcal{N} \models \varphi'[b_1, \dots, b_n].$$

Take an approximation σ' of σ such that $\sigma < \sigma' < \tau$. By the definition of Σ, we have $\operatorname{neg}(\sigma') \in \Sigma$ and therefore $\mathcal{M} \models \operatorname{neg}(\sigma')[a_1, \dots, a_n]$; but this contradicts the fact that $\sigma \in p$ (and the definition of p), so the claim is proved.

By the preceding claim and the Compactness Theorem, we find formulas $\chi_1, \dots, \chi_k \in \Sigma$ such that whenever $\mathcal{M}$ is an approximate model of Γ and $a_1, \dots, a_n$ are elements of suitable sorts of $\mathcal{M}$ with $\|a_j\| \leq r$ for $j = 1, \dots, n$, we have

$$\mathcal{M} \models \varphi[a_1, \dots, a_n] \quad \text{implies} \quad \mathcal{M} \models \operatorname{neg}(\chi_1) \vee \dots \vee \operatorname{neg}(\chi_k)[a_1, \dots, a_n].$$

Each χ_i is of the form $\operatorname{neg}(\sigma_i)$, where there exists $\tau_i > \sigma_i$ such that whenever $\mathcal{M}$ is an approximate model of Γ and $a_1, \dots, a_n$ are elements of suitable sorts of $\mathcal{M}$ with $\|a_j\| \leq r$ for $j = 1, \dots, n$,

$$\mathcal{M} \models \tau_i[a_1, \dots, a_n] \quad \text{implies} \quad \mathcal{M} \models \varphi'[a_1, \dots, a_n].$$

Thus, Condition (i) follows by taking as θ the formula $\sigma_1 \vee \dots \vee \sigma_k$ and as θ' the formula $\tau_1 \vee \dots \vee \tau_k$. □

13.13 Definition. A positive bounded theory Γ in a signature L *admits quantifier elimination* if every L-formula is approximable in Γ by quantifier-free formulas.

13.14 Remarks.

(i) Let Γ be a positive bounded theory in a signature L, and let $L(C)$ be an extension of L by constants. If Γ admits quantifier elimination in L then Γ admits quantifier elimination in $L(C)$.

(ii) Let $\Gamma \subseteq \Gamma'$ be positive bounded theories in a signature L. If Γ admits quantifier elimination in L, then Γ' admits quantifier elimination in L.

13.15 Lemma. *Suppose that Γ is a positive bounded theory in a signature L and ψ is a positive bounded L formula. If every formula φ with $\varphi < \psi$ is approximable in Γ by quantifier-free formulas, then* $\operatorname{neg}(\psi)$ *is approximable in Γ by quantifier-free formulas.*

Proof. If $\psi(x_1, \dots, x_n)$ be a positive bounded L-formula, a typical approximation of $\operatorname{neg}(\psi)$ is a positive bounded formula of the form $\operatorname{neg}(\varphi)$, where $\varphi < \psi$. Assume that such φ is approximable in Γ by quantifier-free formulas and take $r > 0$ and an approximation φ' of φ with $\varphi < \varphi' < \psi$. Fix a quantifier-free positive bounded formula $\theta(x_1, \dots, x_n)$ and an approximation θ' of θ such that whenever $\mathcal{M}$ is an approximate model of Γ and $a_1, \dots, a_n$ are elements of suitable sorts of $\mathcal{M}$ with $\|a_j\| \le r$ for $j = 1, \dots, n$,

$$\mathcal{M} \models \varphi[a_1, \dots, a_n] \quad \text{implies} \quad \mathcal{M} \models \theta[a_1, \dots, a_n],$$
$$\mathcal{M} \models \theta'[a_1, \dots, a_n] \quad \text{implies} \quad \mathcal{M} \models \varphi'[a_1, \dots, a_n].$$

Fix also an approximation θ^* of θ such that $\theta < \theta^* < \theta'$. Then, if $\mathcal{M}$ and $a_1, \dots, a_n$ are as above, we have

$$\mathcal{M} \models \operatorname{neg}(\psi)[a_1, \dots, a_n] \quad \text{implies} \quad \mathcal{M} \models \operatorname{neg}(\theta')[a_1, \dots, a_n]$$
$$\mathcal{M} \models \operatorname{neg}(\theta^*)[a_1, \dots, a_n] \quad \text{implies} \quad \mathcal{M} \models \operatorname{neg}(\varphi)[a_1, \dots, a_n]$$

by the basic properties of neg (see Remark 5.13). □

13.16 Lemma. *Suppose that Γ is a positive bounded theory in a signature L and that every existential positive bounded L-formula with one existential quantifier is approximable in Γ by quantifier-free formulas. Then Γ admits quantifier elimination.*

Proof. Note that by Lemma 13.15, the hypothesis implies that every positive bounded formula in prenex normal form with one universal bounded quantifier is approximable in Γ by quantifier-free formulas. The proof that every positive bounded L-formula φ is approximable in Γ by quantifier-free formulas is by induction on the complexity of φ. □

13.17 Proposition. *Let Γ be a positive bounded theory in a signature L such that the class of approximate models of Γ is uniform. Then the following conditions are equivalent:*

(i) Γ *admits quantifier elimination;*

(ii) *If $\mathcal{M}$ and $\mathcal{N}$ are approximate models of Γ, then every embedding of a substructure of $\mathcal{M}$ into $\mathcal{N}$ can be extended to an embedding of $\mathcal{M}$ into an approximate elementary extension of $\mathcal{N}$.*

Moreover, if $\operatorname{card}(L) \leq \kappa$*, then in condition (ii) it suffices to consider approximate models $\mathcal{M}$ such that every sort of $\mathcal{M}$ has density character $\leq \kappa$.*

Proof. Let us first prove that (ii) implies (i). By Lemma 13.16 it suffices to show that each existential L-formula with one existential quantifier is approximable in Γ by quantifier-free formulas. Consider such a formula $\exists_r y\varphi(x_1, \ldots, x_n, y)$ where φ is a quantifier-free positive bounded L-formula and x_j is a variable of sort s_j, for $j = 1, \ldots, n$. We show that this formula satisfies condition (ii) of Proposition 13.12. Therefore, suppose we are given approximate models $\mathcal{M}$ and $\mathcal{N}$ of Γ, substructures $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{N}_0 \subseteq \mathcal{N}$, an isomorphism $(T^{(s)} \mid s \in S)$ of $\mathcal{M}_0$ into $\mathcal{N}_0$, and elements $a_1, \ldots a_n$ of suitable sorts of $\mathcal{M}_0$, such that

$$\mathcal{M} \models_{\mathcal{A}} \exists_r y\varphi(x_1, \ldots, x_n, y)[a_1, \ldots, a_n].$$

We need to prove that

$$\mathcal{N} \models_{\mathcal{A}} \exists_r y\varphi(x_1, \ldots, x_n, y)[T^{(s_1)}(a_1), \ldots, T^{(s_1)}(a_n)].$$

Fix $s > r$ and any approximation φ' of φ, and let b be an element of a suitable sort of $\mathcal{M}$ such that $\|b\| \leq s$ and

$$\mathcal{M} \models \varphi'(x_1, \ldots, x_n, y)[a_1, \ldots, a_n, b].$$

Applying the assumed condition (ii), we may extend T to an embedding of $\mathcal{M}$ into an approximate elementary extension $\mathcal{N}'$ of $\mathcal{N}$. Let c be the image of b under this embedding. This implies that $\|c\| \leq s$ and, since φ is quantifier-free, we also have

$$\mathcal{N}' \models_{\mathcal{A}} \varphi'(x_1, \ldots, x_n, y)[T^{(s_1)}(a_1), \ldots, T^{(s_1)}(a_n), c],$$

and hence

$$\mathcal{N} \models_{\mathcal{A}} \exists_s y\varphi'(x_1, \ldots, x_n, y)[T^{(s_1)}(a_1), \ldots, T^{(s_1)}(a_n)].$$

Since s and φ' were arbitrary, we may conclude that

$$\mathcal{N} \models_{\mathcal{A}} \exists_r y \varphi(x_1, \ldots, x_n, y)[T^{(s_1)}(a_1), \ldots, T^{(s_1)}(a_n)].$$

To prove (i)$\Rightarrow$(ii), assume that Γ admits quantifier elimination, and suppose that $\mathcal{M}$ and $\mathcal{N}$ are approximate models of Γ and $(T^{(s)} \mid s \in S)$ is an isomorphism of a substructure $\mathcal{M}_0$ of $\mathcal{M}$ into $\mathcal{N}$. Then, by the quantifier elimination assumption, whenever $\varphi(x_1, \ldots, x_n)$ is an existential L-formula with x_j of sort s_j for $j = 1, \ldots, n$ and $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}_0$, we have

$$\mathcal{M} \models \varphi[a_1, \ldots, a_n] \quad \text{implies} \quad \mathcal{N} \models_{\mathcal{A}} \varphi[T^{(s_1)}(a_1), \ldots, T^{(s_1)}(a_n)].$$

Therefore (ii) follows by an application of the Compactness Theorem (Proposition 9.22). □

13.18 Example. Fix p in the interval $1 \leq p < \infty$ and let $\mathfrak{C}_p$ be the class of all Banach lattices that are isomorphic to $L_p(\mu)$ for some *atomless* measure μ. There is a set Γ_p of positive bounded sentences in the signature for Banach lattices such that $\mathfrak{C}_p$ is the set of all Banach lattices that are approximate models of Γ_p. The Downward Löwenheim-Skolem Theorem (Proposition 9.13) shows that Γ_p is a complete positive bounded theory, since any two separable members of $\mathfrak{C}_p$ are isomorphic.

We indicate briefly a proof that Γ_p *admits quantifier elimination.* This is done by verifying condition (ii) of Proposition 13.17; for this we use the structure theory of abstract L_p-spaces as found in the proof of Theorem 3 in [BDCK66] and in Section 15 in [Lac74].

Let $\mathcal{M}, \mathcal{N}$ be members of $\mathfrak{C}_p$ and let T be an isomorphism from a substructure $\mathcal{M}_0$ of $\mathcal{M}$ into $\mathcal{N}$. We must extend T to an embedding of $\mathcal{M}$ into some approximate elementary extension of $\mathcal{N}$. Since the signature of Banach lattices is countable, we may assume that $\mathcal{M}$ is separable and it is evidently no loss of generality to assume that $\mathcal{M}$ is complete.

Let $\mathcal{N}'$ be an $\aleph_1$-saturated approximate elementary extension of $\mathcal{N}$. (It suffices to consider the case where $\mathcal{N}'$ is an ultrapower of $L_p(\lambda)$, where λ is Lebesgue measure on the unit interval. This may simplify some of the arguments below for some readers.) We will prove that every embedding T of a substructure of $\mathcal{M}$ into $\mathcal{N}'$ extends to an embedding of $\mathcal{M}$ into $\mathcal{N}'$. Using a Zorn's Lemma argument, it suffices to prove that if T is an embedding of a proper substructure $\mathcal{M}_0$ of $\mathcal{M}$ into $\mathcal{N}'$, then there exists *some* $\mathcal{M}_1$ with $\mathcal{M}_0 \subset \mathcal{M}_1 \subseteq \mathcal{M}$ and an extension of T to an embedding of $\mathcal{M}_1$ into $\mathcal{N}'$.

So, let $\mathcal{M}_0 \subset \mathcal{M}$ and let T be an embedding of $\mathcal{M}_0$ into $\mathcal{N}'$. First we may assume that $\mathcal{M}_0$ is not $\{0\}$; otherwise, take e to be any positive element of $\mathcal{M}$ having norm 1, and extend T so it takes e to any similar element of $\mathcal{N}'$. The one-dimensional space spanned by e is a substructure (sublattice) of $\mathcal{M}$ and T is an embedding. Next we may assume that $\mathcal{M}_0$ is closed; otherwise take $\mathcal{M}_1$ to be the closure and extend T by uniform continuity.

With these reductions, $\mathcal{M}_0$ is a non-zero closed vector sublattice of $\mathcal{M}$, and therefore it is an $L_p(\nu)$-space in its own right (but the measure ν need not be atomless). By separability, there is a norm-one, positive element e of $\mathcal{M}_0$ which has full support; that is, the band generated by e in $\mathcal{M}_0$ is all of $\mathcal{M}_0$.

Next consider the case where the band generated by e in $\mathcal{M}$ is *not* all of $\mathcal{M}$. Then there must exist a norm-one, positive element f of $\mathcal{M}$ that is orthogonal to e. Since $\mathcal{N}'$ is $\aleph_1$-saturated, we can find a norm-one, positive element f' of $\mathcal{N}'$ which is orthogonal to $T(e)$. In this situation we let $\mathcal{M}_1$ be the substructure of $\mathcal{M}$ that is generated by $\mathcal{M}_0$ and f; our choice of f ensures that $\mathcal{M}_1$ is the Banach lattice direct sum of $\mathcal{M}_0$ and the one dimensional sublattice generated by f. Thus we extend T to an embedding of $\mathcal{M}_1$ into $\mathcal{N}'$ by mapping f to f'.

Finally, we must deal with the case where the band generated by e in $\mathcal{M}$ *is* all of $\mathcal{M}$. In this case we use the structure theory for abstract L_p-spaces to pass to an isomorphic situation in which $\mathcal{M}$ is L_p (where the measure is Lebesgue measure on the unit interval $([0,1],\Sigma,\lambda)$) and $\mathcal{M}_0$ is the sublattice of all functions in L_p that are measurable with respect to Σ_0, where Σ_0 is a sub-σ-algebra of Σ. Let A be a set of positive measure in Σ that is not in Σ_0; such a set exists because $\mathcal{M}_0$ is a proper closed sublattice of $\mathcal{M}$. We want to extend T to an embedding of the sublattice of $\mathcal{M}$ that is generated by $\mathcal{M}_0$ and χ_A (the characteristic function of A).

Again we use the structure theory of abstract L_p-spaces, this time applied to $\mathcal{N}'$. Namely, we may represent $\mathcal{N}'$ as $L_p(\mu)$ for some (necessarily atomless) measure μ, and we may do this in such a way that for every set B in Σ_0, $T(\chi_B)$ is the characteristic function $\chi_{\sigma(B)}$ of some μ-measurable set $\sigma(B)$. It is a routine saturation argument to show that there exists a μ-measurable set A' such that for any $B\in\Sigma_0$, we have $\lambda(B\cap A)=\mu(\sigma(B)\cap A')$ and $\lambda(B\setminus A)=\mu(\sigma(B)\setminus A')$. The desired extension of T can then be taken to map χ_A to $\chi_{A'}$.

We close this example by pointing out that it can be paralleled for Banach spaces without lattice structure using the equimeasurability results of Rudin [Rud76] and Plotkin [Plo71]. The result is that the class of $L_p(\mu)$-spaces, where μ is atomless, admits quantifier elimination *in the language of pure Banach spaces*, as long as $1 \leq p < \infty$ and $p \neq 4, 6, 8, \ldots$. As above, the proof goes via Proposition 13.17, and we need to verify condition (ii); however, in this situation it is linear subspaces and linear embeddings that need to be considered. The results of Rudin and Plotkin are used to extend an embedding from an arbitrary linear subspace to the closed sublattice that it generates, after which the argument given above takes over.

Approximability by Formulas in a Given Class

13.19 Definition. Suppose that L is a signature and Θ is a class of positive bounded L-formulas that is closed under approximations. If Γ is a positive bounded theory in L and $\varphi(x_1, \ldots, x_n)$ is a positive bounded L-formula, we will say that φ is *approximable in* Γ *by formulas from* Θ if the following condition holds. For every approximation φ' of φ and every positive real number r there exists $\theta(x_1, \ldots, x_n) \in \Theta$ and an approximation θ' of θ such that whenever $\mathcal{M}$ is an approximate model of Γ and $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$ with $\|a_j\| \leq r$ for $j = 1, \ldots, n$,

$$\begin{aligned} \mathcal{M} \models \varphi[a_1, \ldots, a_n] \quad &\text{implies} \quad \mathcal{M} \models \theta[a_1, \ldots, a_n], \\ \mathcal{M} \models \theta'[a_1, \ldots, a_n] \quad &\text{implies} \quad \mathcal{M} \models \varphi'[a_1, \ldots, a_n]. \end{aligned}$$

Notice that the concept of approximability by quantifier-free formulas introduced in Definition 13.10 is a particular instance of the concept defined above, namely, when Θ is the class of all quantifier-free L-formulas.

13.20 Proposition. *Suppose that L is a signature and Θ is a class of positive bounded L-formulas that is closed under conjunction, disjunction, and approximations. Let Γ be a positive bounded theory in L such that the class of approximate models of Γ is uniform, and let $\varphi(x_1, \ldots, x_n)$ be a positive bounded L-formula. Then the following conditions are equivalent:*

(i) *Either φ is approximable in Γ by formulas from Θ, or φ is satisfiable in every approximate model of Γ, or φ is satisfiable in no approximate model of Γ;*

(ii) *If $\mathcal{M}$ and $\mathcal{N}$ are approximate models of Γ and $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are elements of suitable sorts of $\mathcal{M}$ and $\mathcal{N}$, respectively, such that for every $\theta(x_1, \ldots, x_n) \in \Theta$,*

$$\mathcal{M} \models_{\mathcal{A}} \theta[a_1, \ldots, a_n] \quad \textit{implies} \quad \mathcal{N} \models_{\mathcal{A}} \theta[b_1, \ldots, b_n],$$

then we have

$$\mathcal{M} \models_{\mathcal{A}} \varphi[a_1, \ldots, a_n] \quad \textit{implies} \quad \mathcal{N} \models_{\mathcal{A}} \varphi[b_1, \ldots, b_n].$$

Proof. Analogous to the proof of Proposition 13.12. □

13.21 Example. As an example of how Proposition 13.20 can be applied, we consider one of the alternative formulations discussed in Chapter 11; the others can be handled similarly.

Let L_0 be the signature for Banach spaces with no additional structure and let Θ be the set of *restricted* positive bounded L_0-formulas. Recall that these are built up from atomic formulas of the form

$$\|k_1x_1 + \cdots + k_nx_n\| \leq r,$$
$$\|k_1x_1 + \cdots + k_nx_n\| \geq r,$$

where $x_1, \ldots, x_n$ are variables of the normed space sort, $k_1, \ldots, k_n$ are positive integers, and r is a rational number, using $\wedge, \vee$ and bounded quantifiers $\exists_r x, \forall_r x$ where r is a positive rational number and x is a variable of the normed space sort.

It is clear that Θ is closed under conjunction, disjunction, and approximations. Moreover, Proposition 11.2 shows that Θ satisfies condition (ii) of Proposition 13.20 when applied to the class of all Banach spaces and to any positive bounded L_0-formula φ. Therefore, every positive bounded L_0-formula is approximable in the class of all Banach spaces by restricted L_0 formulas.

Banach Spaces with Isomorphic Ultrapowers

Here we work in the pure Banach space signature L_0 and we identify every Banach space L_0-structure with its underlying Banach space. (See Chapter 7.)

13.22 Definition. Let φ be a positive bounded L_0-formula. For every rational number $\lambda > 1$ we define an approximation φ_λ of φ. The definition is by induction on the complexity of φ:

If φ is:	φ_λ is:
$\|t\| \le r$	$\|t\| \le \lambda r$
$\|t\| \ge r$	$\|t\| \ge \frac{r}{\lambda}$
$(\psi \wedge \theta)$	$(\psi_\lambda \wedge \theta_\lambda)$
$(\psi \vee \theta)$	$(\psi_\lambda \vee \theta_\lambda)$
$\exists_r x\, \psi$	$\exists_{\lambda r} x\, \psi_\lambda$
$\forall_r x\, \psi$	$\forall_{\frac{r}{\lambda}} x\, \psi_\lambda$

If X and Y are Banach spaces and λ is a real number with $\lambda \ge 1$, we say that X and Y are λ-*isomorphic* if there is a linear isomorphism $T\colon X \to Y$ such that $\|T\|, \|T^{-1}\| \le \lambda$. In this case we say that the map T is a λ-isomorphism.

A concept of approximate elementary equivalence suitably adapted to the "λ-isomorphic" context was developed in [HH86]. Here we summarize briefly what was done in the first part of that paper, and prove a version of the Ultraproduct Theorem (Theorem 10.8) for λ-isomorphisms between ultraproducts of Banach spaces. The corresponding result for ultrapowers of Banach spaces is Theorem 4 in [HH86].

13.23 Remarks. Let φ be a positive bounded L_0-formula. Then:

(i) For each $\lambda > 1$, φ_λ is an approximation of φ;
(ii) for every approximation φ' of φ there exists a rational number $\lambda > 1$ such that $\varphi < \varphi_\lambda < \varphi'$;
(iii) if $\lambda, \mu > 1$, then $(\varphi_\lambda)_\mu$ is $\varphi_{\lambda\mu}$;
(iv) if $\lambda < \mu$, then $(\mathrm{neg}(\varphi_\mu))_\lambda$ is $\mathrm{neg}(\varphi_{\mu/\lambda})$.

By Proposition 6.4 and Remark 13.23(i)(ii), for every pair of Banach spaces X and Y, $X \equiv_{\mathcal{A}} Y$ if and only if for every positive bounded L_0-formula φ we have $X \models \varphi \quad \text{implies} \quad Y \models \varphi_\lambda, \quad \text{for every } \lambda > 1$. It is therefore natural to generalize the concept $\equiv_{\mathcal{A}}$ of approximate elementary equivalence for L_0-structures as follows. (See [HH86, Definition 2 and Lemma 3].)

13.24 Definition. If X, Y are Banach spaces and λ is a real number with $\lambda \ge 1$, we write $X \equiv_{\mathcal{A}}^{\lambda} Y$ if for every positive bounded L_0-sentence φ,

$$X \models \varphi \quad \text{implies} \quad Y \models \varphi_\mu, \quad \text{for every } \mu > \lambda.$$

13.25 Proposition. *If X, Y, and Z are Banach spaces and $\lambda, \mu \geq 1$, then*

(i) $X \equiv_{\mathcal{A}}^{\lambda} X$;
(ii) *If* $X \equiv_{\mathcal{A}}^{\lambda} Y$, *then* $Y \equiv_{\mathcal{A}}^{\lambda} X$;
(iii) *If* $X \equiv_{\mathcal{A}}^{\lambda} Y$ *and* $Y \equiv_{\mathcal{A}}^{\mu} Z$, *then* $X \equiv_{\mathcal{A}}^{\lambda\mu} Z$.

Proof. Part (i) is immediate and Part (ii) follows from Remark 13.23-(ii). To prove (iii), assume that $X \equiv_{\mathcal{A}}^{\lambda} Y$ and φ is a positive bounded L_0-sentence such that $Y \models \varphi$ and $X \models \mathrm{neg}(\varphi_\nu)$ for some $\nu > \lambda$. Then, by Remark 13.23-(iii), for every μ with $1 < \mu < \nu$ we have $Y \models \mathrm{neg}(\varphi_\mu)$; but this contradicts the assumption that $Y \models \varphi$. ☐

The main result of this section is the following version of the Ultraproduct Theorem (Theorem 10.8). (Also, see [HH86, Remark, page 313].)

13.26 Theorem. *Let κ be an infinite cardinal. The ultrafilter D on 2^κ given by Theorem 10.8 satisfies the following property: if λ is a real number with $\lambda \geq 1$ and $(X_\xi \mid \xi < \Lambda)$ and $(Y_\xi \mid \xi < \Lambda)$ are families of Banach spaces such that* density(X_ξ) *and* density(Y_ξ) *are uniformly bounded by κ and*

$$\Big(\prod_{\xi<\Lambda} X_\xi\Big)_D \equiv_{\mathcal{A}}^{\lambda} \Big(\prod_{\xi<\Lambda} Y_\xi\Big)_D,$$

then $(\prod_{\xi<\Lambda} X)_D$ and $(\prod_{\xi<\Lambda} Y)_D$ are λ-isomorphic.

One way to prove this theorem would be to make the appropriate changes in the proof of Theorem 10.8; that is, one would suitably change the approximations of positive bounded formulas in the inductive construction by μ-approximations, for rational numbers $\mu > \lambda$. This approach would have the advantage that one could apply it to signatures more general than L_0. Below we present an alternative proof which invokes Theorem 10.8 as well as the following result proved in [HH86, Theorem 4]. (Note that the argument at [HH86, lines 12–17, page 308] uses a special case of what is given here in Theorem 10.7.)

13.27 Theorem. *Let X and Y be separable Banach spaces and let λ be a real number with $\lambda \geq 1$. The following conditions are equivalent:*

(i) $X \equiv_{\mathcal{A}}^{\lambda} Y$;
(ii) *There exist Banach spaces $\widehat{X}$ and $\widehat{Y}$ such that $X \preceq_{\mathcal{A}} \widehat{X}$ and $Y \preceq_{\mathcal{A}} \widehat{Y}$ and $\widehat{X}, \widehat{Y}$ are λ-isomorphic.*

Proof of Theorem 13.26. Let D be the ultrafilter on 2^κ given by Theorem 10.8, and let

$$X = (\prod_{\xi<\Lambda} X_\xi)_D$$
$$Y = (\prod_{\xi<\Lambda} Y_\xi)_D.$$

The Downward Löwenheim-Skolem Theorem (Proposition 9.13) gives us separable spaces X' and Y' such that $X \equiv_{\mathcal{A}} X'$ and $Y \equiv_{\mathcal{A}} Y'$. Then Theorem 13.27 gives λ-isomorphic Banach spaces $\widehat{X'}$ and $\widehat{Y'}$ such that $X' \preceq_{\mathcal{A}} \widehat{X'}$ and $Y' \preceq_{\mathcal{A}} \widehat{Y'}$. By the Downward Löwenheim-Skolem Theorem (applied to a structure that contains $\widehat{X'}$ and $\widehat{Y'}$ as sorts and the given λ-isomorphism between them as a distinguished function), we obtain separable Banach spaces X'' and Y'' such that $X'' \equiv_{\mathcal{A}} X$ and $Y'' \equiv_{\mathcal{A}} Y$, and a λ-isomorphism $T : X'' \to Y''$. Now $(T)_D : (X'')_D \to (Y'')_D$ is also a λ-isomorphism. By the choice of D, X is isometric to $(X'')_D$ and Y is isometric to $(Y'')_D$. Composing $(T)_D$ with these isometries, we obtain the desired λ-isomorphism between X and Y. $\square$

14
Types

14.1 Definition. Suppose that $\mathcal{M}$ is a normed space L-structure, $L(C)$ is an extension of L by constants, and $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$ (see Definition 3.7). Let $e_1, \ldots, e_n$ be elements of $\mathcal{M}$. Fix distinct variables $x_1, \ldots, x_n$ of L such that x_j and e_j have the same sort for each $j = 1, \ldots, n$. We define the *type of* $(e_1, \ldots, e_n)$ *over* $\mathbf{A}$, denoted

$$\mathrm{tp}_{\mathcal{M}}(e_1, \ldots, e_n/\mathbf{A}),$$

to be the set of all positive bounded $L(C)$-formulas $\varphi(x_1, \ldots, x_n)$ such that

$$(\mathcal{M}, \mathbf{A}) \models_{\mathcal{A}} \varphi[e_1, \ldots, e_n].$$

When it can be done without confusion, we omit the subscript $\mathcal{M}$ from this notation and denote the type by

$$\mathrm{tp}(e_1, \ldots, e_n/\mathbf{A}).$$

If C is empty, we write $\emptyset$ in place of $\mathbf{A}$ in this notation, or omit it entirely.

14.2 Remarks.

(i) Let $\mathcal{M}$, $\mathbf{A}$ be as above, and let $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ be elements of $\mathcal{M}$ such that e_j and e'_j are in the same sort of $\mathcal{M}$ for $j = 1, \ldots, n$. Then

$$\mathrm{tp}_{\mathcal{M}}(e_1, \ldots, e_n/\mathbf{A}) = \mathrm{tp}_{\mathcal{M}}(e'_1, \ldots, e'_n/\mathbf{A})$$

if and only if

$$(\mathcal{M}, \mathbf{A}, e_1, \ldots, e_n) \equiv_{\mathcal{A}} (\mathcal{M}, \mathbf{A}, e'_1, \ldots, e'_n).$$

(ii) If $\mathrm{tp}(e_1,\dots,e_n/\mathbf{A}) = \mathrm{tp}(e'_1,\dots,e'_n/\mathbf{A})$, then $\|e_j\| = \|e'_j\|$, for $j = 1,\dots,n$.

(iii) If $(\mathcal{M},\mathbf{A})$ is an $L(C)$-structure and $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$, then

$$\mathrm{tp}_{\mathcal{M}}(e_1,\dots,e_n/\mathbf{A}) = \mathrm{tp}_{\mathcal{N}}(e_1,\dots,e_n/\mathbf{A}).$$

14.3 Proposition. *Suppose that $\mathcal{M}$ is a normed space L-structure, $L(C)$ is an extension of L by constants, and $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$. If $r_1,\dots,r_n$ are nonnegative rational numbers and $t(x_1,\dots,x_n)$ is a set of $L(C)$-formulas, then the following conditions are equivalent:*

(i) *There exists a structure $\mathcal{N}$ such that $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$ and elements $e_1,\dots,e_n$ of suitable sorts of $\mathcal{N}$ with $\|e_j\| \le r_j$ for $j = 1,\dots,n$ such that*

$$t(x_1,\dots,x_n) = \mathrm{tp}_{\mathcal{N}}(e_1,\dots,e_n/\mathbf{A}).$$

(ii)(a) *The formula $\bigwedge_{j=1}^{n} \|x_j\| \le r_j$ is in $t(x_1,\dots,x_n)$;*

(b) *Every finite subset of t^+ is satisfiable in $(\mathcal{M},\mathbf{A})$;*

(c) *Whenever $\varphi(x_1,\dots,x_n)$ is a positive bounded $L(C)$-formula and φ' is an approximation of φ, we have $\varphi \in t$ or $\mathrm{neg}(\varphi') \in t$.*

Proof. The implication (i)⇒(ii) is immediate from the definitions. To prove (ii)⇒(i) let $B = \{b_1,\dots,b_n\}$ be a set of new constant symbols, where we take the sort of b_j to be the same as the sort of x_j for $j = 1,\dots,n$. Let $\mathfrak{C}$ be the class of $L(C \cup B)$-structures of the form

$$(\mathcal{M},\mathbf{A},\mathbf{B}),$$

where $\mathbf{B}$ is an interpretation of $\{b_1,\dots,b_n\}$ in $(\mathcal{M},\mathbf{A})$, and $\|\mathbf{B}(b_j)\| \le r_j$ for $j = 1,\dots,n$. Then $\mathfrak{C}$ is a uniform class of $L(C \cup B)$-structures. By (ii), every finite subset of $(t(b_1,\dots,b_n))^+$ is satisfied by a structure in $\mathfrak{C}$, so (i) follows from the Compactness Theorem (Proposition 9.5) and Corollary 9.7. □

14.4 Definition. Let $\mathcal{M}$, $L(C)$, and $\mathbf{A}$ be as above. A set $t(x_1,\dots,x_n)$ of positive bounded $L(C)$-formulas satisfying the (equivalent) conditions of Proposition 14.3 is called a *type over* $\mathbf{A}$. If $\mathcal{N}$ and $(e_1,\dots,e_n)$ are as in (i) of the proposition, we say that $(e_1,\dots,e_n)$ *realizes t in* $\mathcal{N}$ or that $(e_1,\dots,e_n)$ is a *realization of t in* $\mathcal{N}$.

14.5 Remarks. Let L, C, $\mathcal{M}$, and $\mathbf{A}$ be as in the statement of Proposition 14.3.

(i) If $t(x_1, \ldots, x_n)$ is a type over $\mathbf{A}$, the structure $\mathcal{N}$ and the elements $e_1, \ldots, e_n$ given by Proposition 14.3 can be taken so that

$$\mathcal{N} \models t[e_1, \ldots, e_n],$$

not just $\mathcal{N} \models_{\mathcal{A}} t[e_1, \ldots, e_n]$); this may be achieved by taking $\mathcal{N}$ to be $\aleph_1$-saturated (see Propositions 9.20 and 9.18).

(ii) Suppose that $(t_i \mid i \in I)$ is a family of types over $\mathbf{A}$ such that t_i is realized in $\mathcal{N}_i$, for every $i \in I$ (so $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}_i$). Using the Compactness Theorem (Proposition 9.5) one can find a structure $\mathcal{N}$ such that $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$ and for every $i \in I$, there is an approximate elementary embedding of $\mathcal{N}_i$ into $\mathcal{N}$ whose restriction to $\mathcal{M}$ is the identity; hence every t_i is realized in $\mathcal{N}$. Thus, any given family of types over $\mathbf{A}$ can be realized in a single approximate elementary extension of $\mathcal{M}$. This observation will be used heavily. (See also Proposition 14.8 below.)

(iii) Suppose that $\mathcal{M}$ is an L-structure and $(t_i \mid i \in I)$ is a family of types over $\mathbf{A}$ as in the previous paragraph. Let κ be an infinite cardinal number such that $\text{card}(L) \leq \kappa$ and the density character of each sort of $\mathcal{M}$ is at most κ. Then, using the Downward Löwenheim-Skolem Theorem (Proposition 9.13), the approximate elementary extension $\mathcal{N}$ of $\mathcal{M}$ in the previous paragraph can be found such that the density character of each sort of $\mathcal{N}$ is at most $\text{card}(I) + \kappa$.

14.6 Definition. If $\Gamma(x_1, \ldots, x_n)$ is a set of L-formulas and Γ_0 is a set of L-formulas such that $\Gamma_0 \subseteq \Gamma$, let us say that Γ is *determined by* Γ_0 (or that Γ_0 *determines* Γ) if the following condition holds: whenever $\mathcal{M}$ is an L-structure and $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$,

$$\mathcal{M} \models_{\mathcal{A}} \Gamma_0[a_1, \ldots, a_n] \qquad \text{implies} \quad \mathcal{M} \models_{\mathcal{A}} \Gamma[a_1, \ldots, a_n].$$

14.7 Proposition. *Suppose that $\mathcal{M}$ is a normed space L-structure and we have the following situation:*

- *$L(C)$ is an extension of L by constants and $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$,*
- *$L(C \cup D)$ is an extension of $L(C)$ by constants and $\mathbf{B}$ is an interpretation of D in $(\mathcal{M}, \mathbf{A})$, and*
- *every element of $\mathcal{M}$ in the range of $\mathbf{B}$ is in the closed L-substructure of $\mathcal{M}$ generated by the range of $\mathbf{A}$.*

Then, whenever $\mathcal{N}$ is an approximate elementary extension of $\mathcal{M}$ and $e_1, \ldots, e_n$ are elements of $\mathcal{N}$, the type $\text{tp}(e_1, \ldots, e_n/\mathbf{A} \cup \mathbf{B})$ *is determined by* $\text{tp}(e_1, \ldots, e_n/\mathbf{A})$.

Proof. By the Perturbation Lemma (Proposition 5.15). □

The language of types allows us to rephrase the concept of saturation.

14.8 Proposition. *Suppose $\mathcal{M}$ is a normed space L-structure and κ is an infinite cardinal. Then the following conditions are equivalent:*

(i) $\mathcal{M}$ *is κ-saturated;*
(ii) *If $L(C)$ is an extension of L by constants with* $\mathrm{card}(C) < \kappa$ *and $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$, then every type over $\mathbf{A}$ is realized in $\mathcal{M}$.*

Spaces of Types

In this section, the signature L will be fixed, and $\mathcal{M}$ will denote a fixed normed space L-structure. Throughout the section, $L(C)$ will denote an extension of L by constants and $\mathbf{M}$ will be an interpretation of C in $\mathcal{M}$ such that every element of $\mathcal{M}$ is in the range of $\mathbf{M}$. The letters $\mathbf{A}$, $\mathbf{B}$, etc. will denote restrictions of $\mathbf{M}$ to subsets of C. We write $\mathbf{A} \subseteq \mathbf{B}$ to indicate that $\mathbf{A}$ is a restriction of $\mathbf{B}$.

Given $\mathbf{A} \subseteq \mathbf{M}$ as above, the set of types over $\mathbf{A}$ is denoted $S(\mathbf{A})$. If $x_1, \dots, x_n$ are distinct variables, the set of types $t(x_1, \dots, x_n)$ over $\mathbf{A}$ will be denoted $S_{(x_1,\dots,x_n)}(\mathbf{A})$. The word "type" will mean "type over $\mathbf{A}$, where $\mathbf{A} \subseteq \mathbf{M}$". Note that $S(\mathbf{A})$ and $S_{(x_1,\dots,x_n)}(\mathbf{A})$ depend only on $\mathrm{Th}_{\mathcal{A}}(\mathcal{M}, \mathbf{A})$.

If $t(x_1, \dots, x_n)$ is a type, we denote by $\|t\|$ the real number $\max_j \|e_j\|$, where $(e_1, \dots e_n)$ is a realization of t. (Note that $\|t\|$ depends only on t and not on the particular realization chosen.)

The Logic Topology on Types

If $\varphi(x_1, \dots, x_n)$ is an $L(C)$-formula, we let

$$[\varphi] = \{\ t(x_1, \dots, x_n) \ \mid\ t \text{ is a type and } \varphi \in t\ \}.$$

14.9 Definition. The *logic topology* on types is defined as follows. If t is a type, the basic neighborhoods of t are the sets of the form $[\varphi]$ with $\varphi \in t^+$. If $x_1, \dots, x_n$ are variables, then the *logic topology on* $S_{(x_1,\dots,x_n)}(\mathbf{A})$ is the restriction to $S_{(x_1,\dots,x_m)}(\mathbf{A})$ of the logic topology.

Note that the logic topology is Hausdorff. Each basic neighborhood $[\varphi]$ is closed in the logic topology; indeed, if t is a type not in $[\varphi]$, then for some $\varphi' > \varphi$ we have that $t \in [\mathrm{neg}(\varphi')]$ while $[\varphi]$ and $[\mathrm{neg}(\varphi')]$ are always disjoint.

14.10 Proposition. *For any distinct variables $x_1, \ldots, x_n$ and any real number $r > 0$, the set of types $t \in S_{(x_1,\ldots,x_n)}(\mathbf{A})$ such that $\|t\| \leq r$ is compact with respect to the logic topology.*

Proof. Proposition 14.10 is in fact a restatement of the Compactness Theorem. Below we present an alternative proof using ultrapowers.

Suppose that $(t_\xi)_{\xi\in\Lambda}$ is a family of types in $S_{(x_1,\ldots,x_n)}(\mathbf{A})$ such that $\|t_\xi\| \leq r$ for every $\xi \in \Lambda$ and that D is an ultrafilter on Λ. Let $\bar{e}_\xi$ be a realization of t_ξ in an approximate elementary extension $\mathcal{N}_\xi$ of $(\mathcal{M}, \mathbf{A})$. Then $(\mathcal{N}_\xi \mid \xi \in \Lambda)$ is a uniform family of structures and $(\bar{e}_\xi)_{\xi\in\Lambda} \in \ell_\infty(\Lambda, \mathcal{N}_\xi \mid \xi \in \Lambda))$. It follows that

$$\operatorname{tp}\big(\,((\bar{e}_\xi)_{\xi\in\Lambda})_D \,/\, \mathbf{A}\,\big) = \lim_{\xi,D}\,(t_\xi)_{\xi\in\Lambda}$$

using Proposition 9.2. □

14.11 Corollary. *The logic topology on $S_{(x_1,\ldots,x_n)}(\mathbf{A})$ is locally compact and σ-compact.*

Proof. For every type $t(x_1, \ldots, x_n) \in S_{(x_1,\ldots,x_n)}(\mathbf{A})$, there exists a rational number r such that the formula $\bigwedge_{j=1}^{n} \|x_j\| \leq r$ is in t^+. By the preceding proposition $[\bigwedge_{j=1}^{n} \|x_j\| \leq r]$ is a compact neighborhood of t. □

14.12 Remark. The logic topology on $S_{(x_1,\ldots,x_n)}(\mathbf{A})$ is not compact $(n > 0)$.

The Metric d on Types

Let $\mathcal{K}$ be an approximate elementary extension of $\mathcal{M}$ such that each type in $S(\mathbf{A})$ is realized in $\mathcal{K}$. (See Remark 14.5(ii) for the existence of $\mathcal{K}$.) Define d on $S_{(x_1,\ldots,x_n)}(\mathbf{A})$ as follows.

$$d(t,t') = \inf\big\{\ \max_j \|e_j - e'_j\| \ \ \big|\ \ \mathcal{K} \models_{\mathcal{A}} t[\bar{e}],\ \ \mathcal{K} \models_{\mathcal{A}} t'[\bar{e}']\ \big\}.$$

It follows easily that d is a pseudometric. The fact that d is a metric follows from the Perturbation Lemma. Note that d does not depend on $\mathcal{K}$, since $\mathcal{K}$ realizes every type of a $2n$-tuple $(e_1, \ldots, e_n, e'_1, \ldots, e'_n)$ over $\mathbf{A}$.

Note that if $t, t' \in S_{(x_1,\ldots,x_n)}(\mathbf{A})$, then by the Compactness Theorem there exist realizations $(e_1, \ldots, e_n)$ of t and $(e'_1, \ldots, e'_n)$ of t' in the same approximate elementary extension of $\mathcal{M}$, such that $\max_j \|e_j - e'_j\| = d(t, t')$.

14.13 Proposition. *The d-topology on $S_{(x_1,\dots,x_n)}(\mathbf{A})$ is finer than the logic topology on $S_{(x_1,\dots,x_n)}(\mathbf{A})$.*

Proof. By the Perturbation Lemma (Proposition 5.15). □

14.14 Proposition. *The metric space $(S_{(x_1,\dots,x_n)}(\mathbf{A}), d)$ is complete.*

Proof. Similar to the proof of Proposition 9.21. □

14.15 Proposition. *The following conditions are equivalent for a pair of types $t, t' \in S_{(x_1,\dots,x_n)}(\mathbf{A})$ and a rational $\delta > 0$:*

(i) $d(t,t') \leq \delta$;

(ii) *Every finite subset of the set*

$$t(x_1,\dots,x_n)^+ \cup t'(x'_1,\dots,x'_n)^+ \cup \{ \bigwedge_{j=1}^{n} \|x_j - x'_j\| \leq \epsilon \mid \epsilon > \delta\}$$

is satisfiable in $(\mathcal{M}, \mathbf{A})$.

(iii) *If κ is an infinite cardinal with $\kappa > \operatorname{card}(L(C))$ and $\mathcal{N}$ is a κ-saturated approximate elementary extension of $\mathcal{M}$, then in $\mathcal{N}$ there exist realizations $(e_1,\dots,e_n)$ of t and $(e'_1,\dots,e'_n)$ of t', such that*

$$\max_j \|e_j - e'_j\| \leq \delta;$$

(iv) *For every $L(C)$-formula $\varphi(x_1,\dots,x_n)$, every approximation φ' of φ, and every rational $\epsilon > \delta$,*

$$\forall_\epsilon z_1 \dots \forall_\epsilon z_n\, \varphi(x_1 + z_1,\dots,x_n + z_n) \in t$$

implies

$$\forall_{\epsilon-\delta} z_1 \dots \forall_{\epsilon-\delta} z_n\, \varphi'(x_1 + z_1,\dots,x_n + z_n) \in t'.$$

Proof. Each implication between two of the conditions follows quickly from the definitions and (occasionally) the Compactness Theorem. □

14.16 Remark. In general there are many important topologies on spaces of types in the functional analysis setting; the more useful ones are given by uniform structures defined using positive bounded formulas. A general formulation of these topologies and preliminary discussion of their properties can be found in [Iov99a] and [Iov96].

Quantifier-Free Types

14.17 Definition. Suppose that $\mathcal{M}$ is a normed space L-structure, $L(C)$ is an extension of L by constants, and $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$. Let $e_1, \ldots, e_n$ be elements of $\mathcal{M}$, and fix distinct variables $x_1, \ldots, x_n$ of L such that x_j and e_j have the same sort for each $j = 1, \ldots, n$. We define the *type of* $(e_1, \ldots, e_n)$ *over* $\mathbf{A}$, denoted

$$\mathrm{tpqf}_{\mathcal{M}}(e_1, \ldots, e_n/\mathbf{A}),$$

to be the set of all quantifier-free positive bounded $L(C)$-formulas $\varphi(x_1, \ldots, x_n)$ such that

$$(\mathcal{M}, \mathbf{A}) \models_{\mathcal{A}} \varphi[e_1, \ldots, e_n].$$

When it can be done without confusion, we omit the subscript $\mathcal{M}$ from this notation and denote the type by

$$\mathrm{tpqf}(e_1, \ldots, e_n/\mathbf{A}).$$

If C is empty, we write $\emptyset$ in place of $\mathbf{A}$ in this notation, or omit it entirely.

14.18 Remarks.

(i) Let $\mathcal{M}$, $\mathbf{A}$ be as above, and let $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ be elements of $\mathcal{M}$ such that e_j and e'_j are in the same sort of $\mathcal{M}$ for all $j = 1, \ldots, n$. Let $\mathcal{M}_0$, $\mathcal{M}'_0$ be the substructures of $\mathcal{M}$ generated by the range of $\mathbf{A}$ and $\{e_1, \ldots, e_n\}$, $\{e'_1, \ldots, e'_n\}$, respectively. Then

$$\mathrm{tpqf}_{\mathcal{M}}(e_1, \ldots, e_n/\mathbf{A}) = \mathrm{tpqf}_{\mathcal{M}}(e'_1, \ldots, e'_n/\mathbf{A})$$

if and only if there is an isomorphism from $\mathcal{M}_0$ onto $\mathcal{M}_1$ that is the identity on the range of $\mathbf{A}$ and takes e_j to e'_j for each $j = 1, \ldots, n$.

(ii) If $\mathrm{tpqf}(e_1, \ldots, e_n/\mathbf{A}) = \mathrm{tpqf}(e'_1, \ldots, e'_n/\mathbf{A})$, then $\|e_j\| = \|e'_j\|$, for $j = 1, \ldots, n$.

(iii) If $(\mathcal{M}, \mathbf{A})$ is an $L(C)$-structure and $\mathcal{M} \subseteq \mathcal{N}$, then

$$\mathrm{tpqf}_{\mathcal{M}}(e_1, \ldots, e_n/\mathbf{A}) = \mathrm{tpqf}_{\mathcal{N}}(e_1, \ldots, e_n/\mathbf{A}).$$

14.19 Proposition. *Suppose that $\mathcal{M}$ is a normed space L-structure, $L(C)$ is an extension of L by constants, and $\mathbf{A}$ is an interpretation of C in $\mathcal{M}$. Let $r_1, \ldots, r_n$ be nonnegative rational numbers and $t(x_1, \ldots, x_n)$ a set of quantifier-free positive bounded $L(C)$-formulas. The following conditions are equivalent:*

(i) *There exists a structure* $\mathcal{N}$ *such that* $\mathcal{M} \preceq_{\mathcal{A}} \mathcal{N}$ *and elements* $e_1, \dots, e_n$ *of suitable sorts of* $\mathcal{N}$ *with* $\|e_j\| \leq r_j$ *for* $j = 1, \dots, n$ *such that*

$$t(x_1, \dots, x_n) = \mathrm{tpqf}_{\mathcal{N}}(e_1, \dots, e_n/\mathbf{A}).$$

(ii)(a) *The formula* $\bigwedge_{j=1}^{n} \|x_j\| \leq r_j$ *is in* $t(x_1, \dots, x_n)$*;*

(b) *Every finite subset of* t^+ *is satisfiable in* $(\mathcal{M}, \mathbf{A})$*;*

(c) *Whenever* $\varphi(x_1, \dots, x_n)$ *is a quantifier-free positive bounded* $L(C)$*-formula and* φ' *is approximation of* φ*, we have* $\varphi \in t$ *or* $\mathrm{neg}(\varphi') \in t$*.*

14.20 Definition. Let $\mathcal{M}$, $L(C)$, and $\mathbf{A}$ be as in the preceding proposition. A set $t(x_1, \dots, x_n)$ of quantifier free $L(C)$-formulas satisfying the (equivalent) conditions in Proposition 14.19 is called a *quantifier-free type* over $\mathbf{A}$. If $\mathcal{N}$ and $(e_1, \dots, e_n)$ are as in (i) of the proposition, we say that $(e_1, \dots, e_n)$ *realizes* t *in* $\mathcal{N}$ or that $(e_1, \dots, e_n)$ is a *realization of* t *in* $\mathcal{N}$.

The following result characterizes quantifier elimination for a complete positive bounded theory in terms of types.

14.21 Proposition. *Suppose that* L *is a signature and* Γ *is a complete positive bounded theory in* L*. The following conditions are equivalent:*

(i) Γ *admits quantifier elimination;*

(ii) *Whenever* $\mathcal{M}$ *is an approximate model of* Γ *and* $e_1, \dots, e_n$ *are elements of* $\mathcal{M}$*, the type*

$$\mathrm{tp}_{\mathcal{M}}(e_1, \dots, e_n/\emptyset)$$

is determined by

$$\mathrm{tpqf}_{\mathcal{M}}(e_1, \dots, e_n/\emptyset).$$

Proof. (i)⇒(ii) follows from the definitions. To prove (ii)⇒(i) one verifies condition (ii) of Proposition 13.12 simultaneously for all positive bounded formulas $\varphi(x_1, \dots, x_n)$. □

Quantifier-Free Types Over Banach Spaces

In this section we work in the pure Banach space signature L_0 and we identify every Banach space L_0-structure with its underlying Banach space. (See Chapter 7.)

The model theoretic concept of *type* was brought into Banach space theory by Krivine and Maurey [KM81]. For analysts, a type over a Banach space X is a function from X into $\mathbb{R}$ that characterizes a possible way to add an element to X while staying inside an ultrapower of X. Namely, given e in an ultrapower of X, one considers the function τ_e defined for $a \in X$ by $\tau_e(a) = \|a + e\|$. Note that τ_e and the norm on X determine the norm on the linear span of X and e. Together with Remark 14.18(i), this makes it clear that as e ranges over an ultrapower of X, there is a 1-1 correspondence between the quantifier-free type $\mathrm{tpqf}(e/X)$ and the function τ_e. (The definition of both objects involves a specific ultrapower of X in the background, from which the elements e are taken and in which norms are calculated.)

From the perspective of model theory, it is natural to generalize this in two ways: first, to consider n-tuples $(e_1, \ldots, e_n)$ in place of single elements e; second, to allow the base space X to be replaced by any subset A of X (including the empty set, in fact). Both of these generalizations yield benefits.

Throughout this section X is a fixed Banach space and A, B denote subsets of X. Let A be such a set, and consider a set $C = \{c_a \mid a \in A\}$ of new constant symbols that is in 1-1 correspondence with A. Define an interpretation $\mathbf{A}$ of C in X by setting $\mathbf{A}(c_a) = a$ for all $a \in A$. Take $e_1, \ldots, e_n$ in any approximate elementary extension of X and consider the quantifier-free type over $\mathbf{A}$ realized by $(e_1, \ldots, e_n)$ over $\mathbf{A}$. These are the objects that will be discussed in this section.

To simplify notation we will ignore the set of constants C and the interpretation $\mathbf{A}$; as is usual in model theory we will write $\mathrm{tpqf}(e_1, \ldots, e_n/A)$ instead of $\mathrm{tpqf}(e_1, \ldots, e_n/\mathbf{A})$ and we will speak of quantifier-free types over A rather than over $\mathbf{A}$. Also, we will write $L(A)$ for $L(C)$ and $(X, a)_{a \in A}$ for the $L(C)$-structure $(X, \mathbf{A})$. In this section, "quantifier-free type" will mean "quantifier-free type over A, realized in an approximate elementary extension of X, where A is a subset of X".

Let A be the closed linear span of B in X. Using Remark 14.18 we see that for any $e_1, \ldots, e_n$ in an approximate elementary extension of X, the quantifier-free type $\mathrm{tpqf}(e_1, \ldots, e_n/A)$ is determined by $\mathrm{tpqf}(e_1, \ldots, e_n/B)$. Hence, we lose no generality by focusing attention on types over closed linear subspaces of X.

14.22 Definition. Let A be a closed subspace of X. A *quantifier-free n-type over A* is a quantifier-free type of the form $\mathrm{tp}(e_1, \ldots, e_n/A)$, where $e_1, \ldots, e_n$ are in some approximate elementary extension of X.

Note that we have chosen to suppress the role of X in this terminology. Also, by Theorem 10.7, any approximate elementary extension of X has an ultrapower that is linearly isometric to an ultrapower of X.

Therefore, every quantifier-free n-type over a subspace of X is realized in some ultrapower of X.

14.23 Proposition. *Let A be a subspace of X. Let $r_1, \ldots, r_n$ be nonnegative rational numbers and $t(x_1, \ldots, x_n)$ a set of quantifier-free positive bounded $L(A)$-formulas. The following conditions are equivalent:*

(i) *There exists a Banach space $\mathfrak{X}$ such that $X \preceq_A \mathfrak{X}$ and elements $e_1, \ldots, e_n$ of $\mathfrak{X}$ with $\|e_j\| \leq r_j$ for $j = 1, \ldots, n$ such that*

$$t(x_1, \ldots, x_n) = \operatorname{tpqf}(e_1, \ldots, e_n/A).$$

(ii)(a) *The formula $\bigwedge_{j=1}^{n} \|x_j\| \leq r_j$ is in $t(x_1, \ldots, x_n)$;*

(b) *Every finite subset of t^+ is satisfiable in $(X, a)_{a \in A}$;*

(c) *Whenever $\varphi(x_1, \ldots, x_n)$ is a quantifier-free positive bounded $L(A)$-formula and φ' is approximation of φ, we have $\varphi \in t$ or $\operatorname{neg}(\varphi') \in t$.*

Proof. This is a special case of Proposition 14.19. □

If A is a subspace of X and $e_1, \ldots, e_n$ are in an approximate elementary extension of X, the quantifier-free type $\operatorname{tpqf}(e_1, \ldots, e_n/A)$ is determined by the truth values of the formulas

$$s_1 \leq \|x + r_1 e_1 + \cdots + r_n e_n\| \leq s_2,$$

as x ranges over A, where $r_1, \ldots, r_n, s_1, s_2$ are rational numbers. Therefore, $\operatorname{tpqf}(e_1, \ldots, e_n/A)$ provides the same information as the function

$$\tau_{e_1, \ldots, e_n, A} \colon A \times \mathbb{R}^n \to \mathbb{R}$$

defined for $a \in A$ and $r_1, \ldots, r_n \in \mathbb{R}$ by

$$\tau_{e_1, \ldots, e_n, A}(a, r_1, \ldots, r_n) = \|a + r_1 e_1 + \cdots + r_n e_n\|.$$

Since A is a subspace, homogeneity of the norm permits us to restrict $\tau_{e_1, \ldots, e_n, A}$ to $A \times \Delta$, where

$$\Delta = \{(r_1, \ldots, r_n) \in \mathbb{R} \mid \sum |r_i| = 1 \text{ and } \sum r_i > 0\},$$

without losing any information.

14.24 Proposition. *Let A be a subspace of X and $\tau \colon A \times \mathbb{R}^n \to \mathbb{R}$ any function. The following conditions are equivalent:*

(i) *There exists a Banach space $\mathfrak{X}$ such that $X \preceq_A \mathfrak{X}$ and elements $e_1, \ldots, e_n$ of $\mathfrak{X}$ such that for all $a \in A$ and $r_1, \ldots, r_n \in \mathbb{R}$*

$$\tau(a, r_1, \ldots, r_n) = \|a + r_1 e_1 + \cdots + r_n e_n\|.$$

(ii) *There exists a family* $(u_1(i), \dots, u_n(i))_{i\in I}$ *in* X^n *and an ultrafilter* D *on* I *such that for all* $a \in A$ *and* $r_1, \dots, r_n \in \mathbb{R}$

$$\tau(a, r_1, \dots, r_n) = \lim_{i,D} \|a + r_1 u_1(i) + \cdots + r_n u_n(i)\|.$$

Proof. (ii)⇒(i) follows from Proposition 14.25-(i) and Proposition 14.10. To prove (i)⇒(ii) fix $e_1, \dots, e_n$ as in (i) and note that, since $e_1, \dots, e_n$ live in an approximate elementary extension of X and $A \subseteq X$, for every formula $\varphi(x_1, \dots, x_n) \in \mathrm{tpqf}(e_1, \dots, e_n/A)$ of the form

$$s_1 \leq \|a + r_1 x_1 + \dots r_n x_n\| \leq s_2$$

(where $a \in A$) and every approximation φ' of φ there exist elements $u_1, \dots, u_n \in X$ such that $X \models \varphi'[u_1, \dots, u_n]$. □

It is natural to define quantifier-free analogs of the topologies on types defined earlier in this chapter. The *quantifier-free logic topology* is the topology on quantifier-free types in which the basic neighborhoods of a quantifier-free type t are the sets of the form $[\varphi]$ with $\varphi \in t^+$. (See Definition 14.9). Let $\mathcal{X}$ be an approximate elementary extension of X in which every quantifier-free type over every subspace of X is realized. The *quantifier-free d-topology* is the topology on quantifier-free types given by the metric d_{qf}, where for two quantifier-free types t, t',

$$d_{\mathrm{qf}}(t, t') = \inf \left\{ \max_j \|e_j - e'_j\| \;\middle|\; \mathcal{X} \models_{\mathcal{A}} t[\bar{e}],\ \ \mathcal{X} \models_{\mathcal{A}} t'[\bar{e}'] \right\}.$$

This definition of the quantifier-free d-topology is independent of $\mathcal{X}$.

14.25 Proposition. *Let* A *be a subspace of* X *and regard quantifier-free* 1*-types over* A *as real-valued functions via the correspondence*

$$\mathrm{tpqf}(e/A) \rightsquigarrow \tau_{e,A}.$$

Then,

(i) *The quantifier-free logic topology corresponds to the topology of pointwise convergence on* A*;*

(ii) *The quantifier-free* d*-topology is finer than the topology of uniform convergence on* A*.*

Proof. (i) is immediate from the definitions; (ii) is a consequence of the triangle inequality. □

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