The Morley rank of a Banach space

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Abstract. We introduce the concepts of Morley rank and Morley degree for structures based on Banach spaces. We characterize ω-stability in terms of Morley rank, and prove the existence of prime models for ω-stable theories.

1. INTRODUCTION

A general framework for the model theoretical analysis of structures based on Banach spaces was introduced in the monograph [5]. It was shown there that the space of types of a Banach space theory is naturally endowed with various uniform topologies on it. A general notion of a uniform structure on the space of types of a complete theory was introduced. In [6], we introduced the concept of stable Banach space theory:

> Let T be a complete Banach space theory, let κ be an infinite cardinal, and let $\mathfrak{U}$ be a uniform structure on the space of types of T. The theory T is *κ-stable* with respect to $\mathfrak{U}$ if for every model E of T of density character less than or equal to κ, the density character of the space of types over E with respect to the uniform topology of $\mathfrak{U}$ is also less than or equal to κ.

In particular, T is ω-stable with respect to $\mathfrak{U}$ if the space of types over every separable model of T is separable with respect to $\mathfrak{U}$. It was shown in [4] that significant classes of Banach space structures (e.g., the spaces L_p, for $1 \leq p < \infty$) are ω-stable with respect to natural, metrizable uniform structures.

We also showed in [6] that when the uniform structure is metrizable, ω-stability implies κ-stability for any infinite cardinal κ, and if T has quantifier elimination, stability is equivalent to ω-stability.

Here, we investigate ω-stability further. We define a Banach space theoretic Morley rank (more precisely, a family of Morley ranks), and characterize ω-stability in terms of it. From this characterization we prove the existence of prime models for theories which are ω-stable with respect to the metric d on the space of types introduced in [5].

The shall assume that the reader is familiar with the basic machinery developed in [5] for the model theoretical analysis of Banach space structures, including the concept of uniform structure on the space of types. However, we shall reprint the definition here for reference.

We extend to this paper the general assumptions and notational conventions of [6]. We deal with Banach space structures, positive bounded formulas, and approximate satisfaction $\models_{\mathcal{A}}$. The terms "formula", "theory" and "type" are used as abbreviations of "positive bounded formula", "positive bounded theory" and "positive bounded type", respectively. If ϕ and ψ are positive bounded formulas, $\phi < \psi$ means that ψ is an approximation of ϕ. The letter T denotes a fixed complete positive bounded theory over a countable language L. All the models considered are models of T. We assume that these models are approximately elementary submodels of some large, saturated model $\mathfrak{C}$. If $\bar{a} \in \mathfrak{C}$, $\|\bar{a}\|$ is an abbreviation of $\max_{1\leq i\leq n}\|a_i\|$. By the $\aleph_0$-saturation of $\mathfrak{C}$, we have $\mathfrak{C} \models_{\mathcal{A}} \phi(\bar{a})$ if and only if $\mathfrak{C} \models \phi(\bar{a})$, for every positive bounded formula ϕ and every $\bar{a} \in \mathfrak{C}$. We write simply $\models \phi(\bar{a})$, omitting $\mathfrak{C}$.

We consider only complete, positive bounded types which are consistent with T. The norm of an n-type p, denoted $\|p\|$, is the norm of any n-tuple realizing p. If A is a subset of a model E, we denote by $L(A)$ the result of expanding the language L with constants and appropriate norm bounds for the elements of A, and $T(A)$ is the theory of $\mathfrak{C}$ in $L(A)$. The set of n-types over A is denoted $S_n(A)$, and $S(A) = \bigcup_{n<\omega} S_n(A)$.

2. SEPARATING PAIRS OF FORMULAS

Let $S(T) = \bigcup_{A\subseteq\mathfrak{C}} S(A)$. The models of T come equipped with a topology, namely, the norm topology. This topology can be transfered onto $S(T)$ as follows. If $p(\bar{x}), q(\bar{x}) \in S_n(A)$, we let

$$d(p,q) = \inf\{\, \|\bar{b}-\bar{c}\| \;:\; \mathfrak{C} \models p(\bar{b}), q(\bar{c}) \,\}.$$

Then, d is a metric on $S_n(A)$. We extend d to all of $S(T)$ by letting $d(p,q) = \infty$ when p and q are types in different sets of variables or over different sets of parameters.

For each rational $\delta > 0$, let $U_\delta = \{\, (p,q) \;:\; d(p,q) \leq \delta \,\}$. The family $\mathfrak{U} = \{\, U_\delta \;:\; \delta \in \mathbb{Q}^+ \,\}$ is an example of a *uniform structure* on $S(T)$:

DEFINITION. A *uniform structure on on the space of types* of T is a family $\mathfrak{U}$ of subsets of $S(T) \times S(T)$, called *vicinities*, such that

(1) $\mathfrak{U}$ is a base for a Hausdorff uniform structure (in the standard topological sense. See, for example, [7, Chapter 6]) on $S(T)$.
(2) For every vicinity U there exists a set of pairs of formulas $\mathcal{D}(U)$ such that
  (i) If $(\phi,\phi') \in \mathcal{D}(U)$, then $\phi < \phi'$.
  (ii) $\mathcal{D}(U)$ *defines* U in the following sense:

  The pair $(p(\bar{x}), q(\bar{x}))$ is in $U \cap S(A) \times S(A)$ if and only if for every pair $(\phi(\bar{x},\bar{y}), \phi'(\bar{x},\bar{y})) \in \mathcal{D}(U)$ and every $\bar{a} \in \mathbb{Q}A$ (= rational multiples of A),

$$\phi(\bar{x},\bar{a}) \in p \quad \text{implies} \quad \phi'(\bar{x},\bar{a}) \in q,$$
$$\phi(\bar{x},\bar{a}) \in q \quad \text{implies} \quad \phi'(\bar{x},\bar{a}) \in p.$$

(iii) There exists a vicinity V with the following property: If $(\phi, \phi') \in \mathcal{D}(U)$ and $\phi'' > \phi'$, there exists $(\psi, \psi') \in \mathcal{D}(V)$ such that $\phi < \psi$ and $\psi' < \phi''$.

In the case of the metric d defined above, we let $\mathcal{D}(U_\delta)$ be the set of pairs (ϕ, ϕ') such that $\phi(\bar{x}, \bar{y})$ is of the form

$$\forall \bar{z}(\|\bar{z}\| \leq \epsilon \rightarrow \sigma(\bar{x} + \bar{z}, \bar{y})),$$

for $\epsilon > \delta$, and $\phi'(\bar{x}, \bar{y})$ is of the form

$$\forall \bar{z}(\|\bar{z}\| \leq \epsilon - \delta \rightarrow \sigma'(\bar{x} + \bar{z}, \bar{y})),$$

with $\sigma' > \sigma$.

Another important example of a metrizable uniform structure on the space of types of T is given by the *Banach-Mazur* metric on types: For $\epsilon \geq 0$, let U_ϵ be the set of pairs of types (p, q) such that there is a $1 + \epsilon$-automorphism $\mathfrak{E}$ mapping p onto q (a linear isomorphism f is a $1 + \epsilon$*-isomorphism* if $\|f\|, \|f^{-1}\| \leq 1 + \epsilon$). The family $\{ U_\epsilon : \epsilon \in \mathbb{Q}^+ \}$ is a metrizable uniform structure on the space of types of T. No theory can be ω-stable with respect to this uniform structure. However, if $\kappa \geq 2^{\aleph_0}$, κ-stability with respect to the metric d implies κ-stability with respect to the Banach-Mazur metric. See [6] for the details, and for a discussion of further natural uniform structures on the space of types.

Hereafter, $\mathfrak{U}$ will denote a fixed uniform structure on the set of types of T, and all the vicinities mentioned are vicinities of $\mathfrak{U}$.

A notion that recurrently underlies the main arguments of [6] is that of a *separating pair of formulas.*

DEFINITION. Let U be a vicinity of $\mathfrak{U}$. A pair of positive bounded formulas $\{ \psi_1(\bar{x}), \psi_2(\bar{x}) \}$ is *U-separating* if $(p(\bar{x}), q(\bar{x})) \notin U$ whenever $\psi_1(\bar{x}) \in p(\bar{x})$ and $\psi_2(\bar{x}) \in q(\bar{x})$.

If Σ is a set of positive bounded formulas, Σ_+ denotes the set

$$\{ \sigma' : \sigma' > \sigma \text{ for some } \sigma \in \Sigma \}.$$

If ϕ is a positive bounded formula, we write ϕ_+ instead of $\{\phi\}_+$.

REMARKS.

(1) If $(p(\bar{x}), q(\bar{x})) \notin U$, there exists a U-separating pair $\{ \psi_1(\bar{x}), \psi_2(\bar{x}) \}$ such that $\psi_1(\bar{x}) \in p(\bar{x})$ and $\psi_2(\bar{x}) \in q(\bar{x})$.

(2) Let V correspond to U as in (2-iii) of the definition of uniform structure. Then, if $(p(\bar{x}), q(\bar{x})) \notin U$, there exists a V-separating pair $\{ \psi_1(\bar{x}), \psi_2(\bar{x}) \}$ such that $\psi_1(\bar{x}) \in p_+(\bar{x})$ and $\psi_2(\bar{x}) \in q_+(\bar{x})$.

DEFINITION. Let $\phi(\bar{x})$ be a positive bounded formula and let U be a vicinity. The U-*Morley rank* of ϕ, denoted $\mathrm{MR}[\phi, U]$, is either an ordinal or ∞. We define $\mathrm{MR}[\phi, U]$ by specifying, for each ordinal α, when $\mathrm{MR}[\phi, U] \geq \alpha$:

- $\mathrm{MR}[\phi, U] \geq 0$ if ϕ is consistent (with T).
- If α is a limit ordinal, $\mathrm{MR}[\phi, U] \geq \alpha$ if $\mathrm{MR}[\phi, U] \geq \gamma$ for every $\gamma < \alpha$.
- If $\alpha = \beta + 1$, $\mathrm{MR}[\phi, U] \geq \alpha$ if there exist positive bounded formulas ψ_i, $0 < i < \omega$, such that

  (i) $\mathrm{MR}[\phi \wedge \psi_i, U] \geq \beta$, for each $i < \omega$;
  (ii) The pair $\{\psi_i, \psi_j\}$ is U-separating for $i < j < \omega$.

- $\mathrm{MR}[\phi, U] = \infty$ if $\mathrm{MR}[\phi, U] \geq \alpha$ for every ordinal α.

For a set of formulas Σ, we define

$$\mathrm{MR}[\Sigma, U] = \min\{\, \mathrm{MR}[\phi, U] \;:\; \phi \text{ is a finite conjunction of formulas in } \Sigma_+ \,\}.$$

REMARK. If $U \subseteq V$ and $\phi \models_{\mathcal{A}} \psi$, then $\mathrm{MR}[\phi, V] \leq \mathrm{MR}[\psi, U]$. In particular, $\mathrm{MR}[\phi \wedge \sigma, U] \leq \mathrm{MR}[\phi, U]$ for any formula σ, and $\mathrm{MR}[\phi, U] \leq \mathrm{MR}[\phi', U]$ for every $\phi' > \phi$.

EXAMPLES.

(1) Let T is the complete theory of a finite dimensional Banach space, and let $\mathfrak{U}$ be the uniform structure of the metric d. Then $\mathrm{MR}[\,\|\bar{x}\| \leq 1, U] = 0$ for every vicinity U by the compactness of the unit ball.

(2) Let $\mathfrak{U}$ be the uniform structure of the Banach-Mazur metric. As we have recalled above, ω-stability with respect to $\mathfrak{U}$ never occurs. Thus, $\mathrm{MR}[\,\|\bar{x}\| \leq 1, U] = \infty$ for every vicinity U by Theorem 9.

(3) Let T be the theory of c_0. Let e_k be the k-th unit vector of c_o, and let

$$a_k = e_0 + \cdots + e_k.$$

Then

$$\|e_k + a_l\| = \begin{cases} 2, & \text{if } k \leq l \\ 1, & \text{if } k > l. \end{cases}$$

(In the notation of [6], the positive bounded formulas

$$\theta(x, y; u, v) : \|y + u\| \geq 2, \qquad \theta'(x, y; u, v) : \|y + u\| \geq 1$$

order the sequence $(a_k * e_k)_{k<\omega}$ in c_0.) This implies T is unstable (with respect to every uniform structure). Again, by Theorem 9, $\mathrm{MR}[\,\|\bar{x}\| \leq 1, U] = \infty$ for every vicinity U of every uniform structure.

(4) The Morley rank is not preserved under distortion of the norm, even by small amounts. For $\epsilon > 0$, define a new norm $\| \|_\epsilon$ on ℓ_2 as follows. For $x = (x_i)_{i\in\omega}$ in ℓ_2, let

$$\|x\|_\epsilon = \|x\| + \epsilon \sup\{\, |x_{2i} + x_{2j-1}| \,:\, i < j < \omega \,\}$$

Then, if e_k is the k-th unit vector, we have

$$\|e_{2k} + e_{2l-1}\|_\epsilon = \begin{cases} \sqrt{2} + 2\epsilon, & \text{if } k < l \\ \sqrt{2} + \epsilon, & \text{if } k > l. \end{cases}$$

(This example was taken from [1], where the credit is given to B. Bollobás. For a similar example, see [8].) As above, ℓ_2 with the new norm is unstable, and the Morley rank of every formula and every vicinity (of every uniform structure) is ∞.

The following lemma will allow us to extend sets of positive bounded formulas without decreasing the U-Morley rank.

LEMMA 1. *If $\phi(\bar{x})$ and $\sigma(\bar{x})$ are formulas, then either*

$$\mathrm{MR}[\phi \wedge \sigma', U] = \mathrm{MR}[\phi, U] \qquad \textit{for all } \sigma' > \sigma,$$

or else

$$\mathrm{MR}[\phi \wedge neg(\sigma'), U] = \mathrm{MR}[\phi, U] \qquad \textit{for some } \sigma' > \sigma.$$

PROOF: We prove by induction on α that if $\mathrm{MR}[\phi, U] \geq \alpha$, then $\mathrm{MR}[\phi \wedge \sigma', U] \geq \alpha$ for all $\sigma' > \sigma$, or $\mathrm{MR}[\phi \wedge neg(\sigma'), U] \geq \alpha$ for some $\sigma' > \sigma$.

The case $\alpha = 0$ is trivial. Suppose $\alpha = \beta + 1$, and let $\{\, \psi_i : i < \omega \,\}$ correspond to ϕ and U as in the definition of Morley rank. We consider two separate cases:

Case 1. There exist $\sigma' > \sigma$ and an infinite subset $X \subseteq \{\, \psi_i : i < \omega \,\}$ such that $\mathrm{MR}[\phi \wedge \psi_i \wedge neg(\sigma'), U] \geq \beta$ for every $i \in X$. In this case, we have $\mathrm{MR}[\phi \wedge neg(\sigma'), U] \geq \alpha$ by definition.

Case 2. Case 1 does not hold. Fix $\sigma' > \sigma$. By the induction hypothesis applied to each of the formulas $\phi \wedge \psi_i \wedge \sigma'$, there exists an infinite $X_{\sigma'} \subseteq \{\, \psi_i : i < \omega \,\}$ such that $\mathrm{MR}[\phi \wedge \psi_i \wedge \sigma'', U] \geq \beta$ for every $i \in X_{\sigma'}$ and $\sigma'' > \sigma'$. Since σ' is arbitrarily close to σ, this means that $\mathrm{MR}[\phi \wedge \sigma'', U] \geq \alpha$ for every $\sigma'' > \sigma$.

Now suppose that α is a limit ordinal. Again, we consider two cases.

Case 1. There exist $\gamma < \alpha$ and $\sigma' > \sigma$ such that $\mathrm{MR}[\phi \wedge \sigma', U] < \gamma$. Take $\sigma < \tau < \sigma'$. By the induction hypothesis, for every $\gamma < \gamma' < \alpha$ there exists $\tau_{\gamma'} > \tau$ such that $\mathrm{MR}[\phi \wedge neg(\tau_{\gamma'}), U] \geq \gamma'$. Since $neg(\tau_{\gamma'}) \models_{\mathcal{A}} neg(\tau)$, we have $\mathrm{MR}[\phi \wedge neg(\tau), U] \geq \gamma'$ for every $\gamma < \gamma' < \alpha$. Hence $\mathrm{MR}[\phi \wedge neg(\tau), U] \geq \alpha$.

Case 2. Case 1 does not hold. Then $\mathrm{MR}[\phi \wedge \sigma', U] \geq \gamma$ for every $\gamma < \alpha$ and every $\sigma' > \sigma$. But then, by definition, $\mathrm{MR}[\phi \wedge \sigma', U] \geq \alpha$ for every $\sigma' > \sigma$. ∎

If ϕ and ψ are formulas with $\phi < \psi$, $[\phi, \psi)$ denotes the set

$$\{\, \sigma \,:\, \phi \leq \sigma < \psi \,\}.$$

The *order topology* on the language L is defined as follows. The neighborhoods of a formula ϕ are the sets $[\phi, \psi)$, where $\phi < \psi$. The following concept was introduced in [5].

DEFINITION. A *quasi-type* over A is a set of positive bounded $L(A)$-formulas whose closure with respect to the order topology is a type.

Clearly, if p is a type, p_+ is a quasi-type.

PROPOSITION 2. *Let $\Gamma(\bar{x})$ be a set of positive bounded $L(A)$-formulas such that*

(1) *Γ is consistent;*
(2) *For some $N > 0$, the formula $\|\bar{x}\| \leq N$ is in Γ;*
(3) *For every positive bounded $L(A)$-formula $\phi(\bar{x})$, either ϕ or $neg(\phi)$ is in Γ.*

Then Γ is a quasi-type over A.

PROOF: See [5].

PROPOSITION 3. *Let U be a vicinity. Let $\Sigma(\bar{x})$ be a consistent set of $L(A)$-formulas, such that for some $N > 0$, the formula $\|\bar{x}\| \leq N$ is in Σ. Then Σ can be extended to a set of formulas $\Gamma(\bar{x})$, such that*

(1) *Γ is a quasi-type over A;*
(2) *Γ is closed under approximations;*
(3) $\mathrm{MR}[\Gamma, U] = \mathrm{MR}[\Sigma, U]$.

PROOF: Let $\phi(\bar{x})$ be a conjunction of formulas in Σ of minimal U-Morley rank. Let $\{\ \sigma_i(\bar{x})\ :\ 1 < i < \mathrm{card}(\ L(A)\ )\ \}$ be a list of all the $L(A)$-formulas in the variables $\bar{x}$. Define

$$\Gamma_0 = \Sigma$$

$$\Gamma_i = \bigcup_{j<i} \Gamma_j \qquad \text{if } i \text{ is limit ordinal,}$$

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\sigma\}_+, & \text{if } \mathrm{MR}[\phi \wedge \sigma', U] = \mathrm{MR}[\phi, U] \text{ for every } \sigma' > \sigma \\ \Gamma_i \cup \{neg(\sigma')\}, & \text{if } \mathrm{MR}[\phi \wedge neg(\sigma'), U] = \mathrm{MR}[\phi, U]. \end{cases}$$

The set $\Gamma = \bigcup_{j<i} \Gamma_i$ suits the bill. ■

COROLLARY 4. *If $\Sigma(\bar{x})$ is a consistent set of $L(A)$-formulas, then there exists a type $p(\bar{x}) \in S(A)$ extending Σ such that* $\mathrm{MR}[p, U] = \mathrm{MR}[\Sigma, U]$.

PROOF: Let Γ correspond to Σ as in Proposition 3, and let p be the closure of Γ in the order topology. Since Γ is closed under approximations, we have $p_+ \subseteq \Gamma$. Hence

$$\mathrm{MR}[p, U] = \mathrm{MR}[p_+, U] \geq \mathrm{MR}[\Gamma, U] = \mathrm{MR}[\Sigma, U]. \ ■$$

3. DEFINABLE SEPARATION

PROPOSITION 5. *For every vicinity V there exists a vicinity $W \subseteq V$ with the following property. If $(p,q) \notin V$, there exists formulas $\sigma \in p_+$ and $\tilde{\sigma} \in q_+$, and formulas $\tau > \sigma$, $\tilde{\tau} > \tilde{\sigma}$ such that*

1. *$\{\sigma, neg(\tau)\}$ is W-separating;*
2. *$\{\tilde{\sigma}, neg(\tilde{\tau})\}$ is W-separating;*
3. *$\tau \wedge \tilde{\tau}$ is inconsistent.*

PROOF: Take a vicinity V^* such that $V^* \circ V^* \subseteq V$. Take also a vicinity W which corresponds to V^* as V corresponds to U in (2-iii) of the definition of uniform structure.

For a type $p(\bar{x}) \in S(A)$, let

$$\Sigma_p^1(\bar{x}) = \{\, \psi'(\bar{x},\bar{a}) \;:\; (\, \psi(\bar{x},\bar{y}), \psi'(\bar{x},\bar{y})\,) \in \mathcal{D}(W),\ \psi(\bar{x},\bar{a}) \in p_+(\bar{x}), \bar{a} \in \mathbb{Q}A \,\}$$
$$\Sigma_p^2(\bar{x}) = \{\, neg(\psi(\bar{x},\bar{a})) \;:\; (\, \psi(\bar{x},\bar{y}), \psi'(\bar{x},\bar{y})\,) \in \mathcal{D}(W),\ neg(\psi'(\bar{x},\bar{a})) \in p_+(\bar{x}), \bar{a} \in \mathbb{Q}A \,\}.$$

Define
$$\Sigma_p(\bar{x}) = \Sigma_p^1(\bar{x}) \cup \Sigma_p^2(\bar{x}).$$

In the proof, the formulas τ and $\tilde{\tau}$ will be constructed out of Σ_p and Σ_q, respectively.

STEP 1. $\Sigma_p \subseteq p$.

PROOF: Clearly, $\Sigma_p^1 \subseteq p$ and $\Sigma_p^2 \subseteq p$.$\square$

STEP 2. *If $\tau(\bar{x}) \in \left(\Sigma_p^1(\bar{x})\right)_+$, there exists $\sigma < \tau$ in p_+ such that $\{\sigma, neg(\tau)\}$ is W-separating.*

PROOF: Take $\tau \in \left(\Sigma_p^1\right)_+$. Then there exist $(\psi(\bar{x},\bar{y}), \psi'(\bar{x},\bar{y})) \in \mathcal{D}(W)$ and $\bar{a} \in \mathbb{Q}A$ such that $\psi(\bar{x},\bar{a}) \in p_+(\bar{x})$ and $\tau > \psi'(\bar{x},\bar{a})$. In this case, we take $\sigma = \psi(\bar{x},\bar{a})$.$\square$

STEP 3. *If $\tau(\bar{x}) \in \left(\Sigma_p^2(\bar{x})\right)_+$, there exists $\sigma < \tau$ in p_+ such that $\{\sigma, neg(\tau)\}$ is W-separating.*

PROOF: Take $\tau(\bar{x}) \in \left(\Sigma_p^2\right)_+$. Then there exist $(\psi(\bar{x},\bar{y}), \psi'(\bar{x},\bar{y})) \in \mathcal{D}(W)$ and $\bar{a} \in \mathbb{Q}A$ such that $neg(\psi'(\bar{x},\bar{a})) \in p_+$ and $\tau > neg(\psi(\bar{x},\bar{a}))$. Thus

$$(*) \qquad neg(\tau) < \psi(\bar{x},\bar{a}).$$

Let σ be a formula of p_+ such that $\sigma < neg(\psi'(\bar{x},\bar{a}))$. First, we notice that $\sigma \wedge \psi'(\bar{x},\bar{a})$ is inconsistent. Indeed, if q is a type containing σ, then $neg(\sigma') \notin q$ for every $\sigma' > \sigma$. In particular, $neg(neg(\psi'(\bar{x},\bar{a}))) \notin q$, i.e., $\psi'(\bar{x},\bar{a}) \notin q$.

We now show that $\{\sigma, neg(\tau)\}$ is W-separating. Let q be a type containing σ, and r a type containing $neg(\tau)$. Then $\psi(\bar{x},\bar{a}) \in r$ (by $(*)$) and $\psi'(\bar{x},\bar{a}) \notin q$ (by the preceding paragraph). Hence $(q,r) \notin W$.$\square$

STEP 4. *If $\tau(\bar{x}) \in (\Sigma_p)_+$, there exists $\sigma < \tau$ in p_+ such that $\{\sigma, neg(\tau)\}$ is W-separating.*

PROOF: This follows from Steps 2 and 3, since $(\Sigma_p)_+ = (\Sigma_p^1)_+ \cup (\Sigma_p^2)_+$.□

STEP 5. *If τ is a conjunction of formulas in $(\Sigma_p)_+$, there exists $\sigma < \tau$ in p_+ such that $\{\sigma, neg(\tau)\}$ is W-separating.*

PROOF: Suppose $\tau = \bigwedge_{i=1}^k \tau_i$, where each τ_i is in $(\Sigma_p)_+$. By Step 4, for each τ_i there exists $\sigma_i < \tau_i$ in p_+ such that $\{\sigma_i, neg(\tau_i)\}$ is W separating. Let σ be $\bigwedge_{i=1}^k \sigma_i$. Then $\sigma < \tau$ and $\{\sigma, neg(\tau)\}$ is W-separating.□

STEP 6. *If $r(\bar{x})$ is a type extending Σ_p, then $(p, r) \in V^*$.*

PROOF: Fix $(\phi(\bar{x}, \bar{y}), \phi'(\bar{x}, \bar{y})) \in \mathcal{D}(V^*)$ and $\bar{a} \in \mathbb{Q}A$. We show that for every $\phi'' > \phi'$,

(I)	$\phi(\bar{x}, \bar{a}) \in p$	implies	$\phi''(\bar{x}, \bar{a}) \in r$
(II)	$\phi(\bar{x}, \bar{a}) \in r$	implies	$\phi''(\bar{x}, \bar{a}) \in p$.

Fix $\phi'' > \phi'$. Find $(\psi, \psi') \in \mathcal{D}(W)$ such that $\phi < \psi$ and $\psi' < \phi''$.

If $\phi(\bar{x}, \bar{a}) \in p$, then $\psi(\bar{x}, \bar{a}) \in p_+$. By definition, $\psi'(\bar{x}, \bar{a}) \in \Sigma_p^1 \subseteq r$. Hence $\phi''(\bar{x}, \bar{a}) \in r$. This proves (I).

If $\phi''(\bar{x}, \bar{a}) \notin p$, then $neg(\phi''(\bar{x}, \bar{a})) \in p$, so $neg(\psi'(\bar{x}, \bar{a})) \in p_+$. By definition, $neg(\psi(\bar{x}, \bar{a})) \in \Sigma_p^2 \subseteq r$. Hence $\phi(\bar{x}, \bar{a}) \notin r$. This proves (II).□

STEP 7. *Let $p(\bar{x})$ and $q(\bar{x})$ be types such that $(p, q) \notin V$. Then $\Sigma_p(\bar{x}) \cup \Sigma_q(\bar{x})$ is inconsistent.*

PROOF: Suppose that $\Sigma_p(\bar{x}) \cup \Sigma_q(\bar{x})$ is consistent, and let $r(\bar{x})$ be a type extending it. By Step 6, $(p, r) \in V^*$ and $(q, r) \in V^*$. But then, $(p, q) \in V^* \circ V^* \subseteq V$, which is a contradiction.□

Now we prove the proposition. Take two types $p(\bar{x})$ and $q(\bar{x})$ such that $(p, q) \notin V$. By Step 7 and the compactness theorem, there exist a conjunction τ of formulas in $(\Sigma_p)_+$ and conjunction $\tilde{\tau}$ of formulas in $(\Sigma_q)_+$, such that $\tau \wedge \tilde{\tau}$ is inconsistent. By Step 5 we can find $\sigma < \tau$ and $\tilde{\sigma} < \tilde{\tau}$ such that $\sigma \in p_+$ and $\tilde{\sigma} \in q_+$, and the pairs $\{\sigma, neg(\tau)\}$ and $\{\tilde{\sigma}, neg(\tilde{\tau})\}$ are W-separating. This is precisely what we wished to prove. ■

4. ω-STABILITY AND MORLEY RANK

The main result of this section is Theorem 9.

LEMMA 6. *For every vicinity V there exists a vicinity W with the following property. If X is an infinite subset of $S_n(A)$ such that $(p,q) \notin V$ whenever p and q are two distinct types in X, then there exist types $p_0, p_1, \dots$ in X, and $L(A)$-formulas $\psi_0, \psi_1, \dots$ such that*

(1) $\psi_i \in (p_i)_+$, *for* $i < \omega$;
(2) *The pair* $\{ \psi_i, \psi_j \}$ *is* W*-separating, for* $i < j < \omega$.

PROOF: Let W correspond to V as in Proposition 5.

For a positive bounded formula θ, let $\mathrm{typ}(\theta)$ denote the set of types over A that contain θ.

We construct, inductively, a sequence of types in X $p_0, p_1, \dots$ sequences of $L(A)$-formulas $\sigma_0 < \tau_0, \sigma_1 < \tau_1$ and infinite subsets of X $X_0 \supseteq X_1 \supseteq \dots$ such that, for each $i < \omega$,

(1) $p_i \in X_i$;
(2) $\sigma_i \in (p_i)_+$;
(3) The pair $\{ \sigma_i, neg(\tau_i) \}$ is W-separating;
(4) $X_{i+1} \subseteq \mathrm{typ}(neg(\tau_i))$.

The lemma then follows by defining, for each $j < \omega$, $\psi_j = \sigma_j \wedge \bigwedge_{i<j} neg(\tau_i)$.

Set $X_0 = X$. Suppose that

$X_0, X_1, \dots, X_i$,
$p_0, p_1, \dots, p_{i-1}$, and
$\sigma_0, \tau_0, \sigma_1, \tau_1, \dots, \sigma_{i-1}, \tau_{i-1}$

have already been defined, in order to define p_i, σ_i, τ_i and X_{i+1}.

Let p and q be two distinct types X_i (these can always be found, for X_i is infinite). Since $(p,q) \notin V$, we can apply Proposition 5 to find $\sigma \in p_+$, $\tilde{\sigma} \in q_+$, $\tau > \sigma$, and $\tilde{\tau} > \tilde{\sigma}$ such that

$\{\sigma, neg(\tau)\}$ is W-separating;
$\{\tilde{\sigma}, neg(\tilde{\tau})\}$ is W-separating;
$\mathrm{typ}(\tau) \cap \mathrm{typ}(\tilde{\tau}) = \emptyset$.

Since X_i is infinite, either $X_i \setminus \mathrm{typ}(\tau)$ or $X_i \setminus \mathrm{typ}(\tilde{\tau})$ is infinite. If $X_i \setminus \mathrm{typ}(\tau)$ is infinite, we let $X_{i+1} = X_i \setminus \mathrm{typ}(\tau)$, $p_i = p$, $\sigma_i = \sigma$, and $\tau_i = \tau$. If $X_i \setminus \mathrm{typ}(\tilde{\tau})$ is infinite, we let $X_{i+1} = X_i \setminus \mathrm{typ}(\tilde{\tau})$, $p_i = q$, $\sigma_i = \tilde{\sigma}$, and $\tau_i = \tilde{\tau}$. The required conditions follow immediately. ■

Let us now introduce some temporary terminology. If U is a vicinity and Y is a set of types, we say that Y is *U-large* if for every positive integer m there exists $Z_m \subseteq Y$ such that

- $\mathrm{card}(Z_m) \geq m$;
- $(p,q) \notin U$ whenever p and q are distinct types in Z_m.

LEMMA 7. *Let U and V be vicinities such that $V \circ V \subseteq U$. If Y is a U-large set of types, there exists an infinite $X \subseteq Y$ such that $(p,q) \notin V$ whenever p and q are distinct types in X.*

PROOF: For $p \in Y$, let $V[p] = \{ q : (p,q) \in V \}$. The proof is based in the following observation: if $p \in Y$ and Z_m is as in the definition of U-large, then $\operatorname{card}(Z_m \cap V[p]) \leq 1$; hence $Y \setminus V[p]$ is U-large.

Let $p_0 \in Y$. The set $Y \setminus V[p_0]$ is large and, in particular, nonempty. Pick $p_1 \in Y \setminus V[p_0]$. The set $Y \setminus (V[p_0] \cup V[p_1])$ is U-large and, in particular, nonempty. Iterating this process ω-many times, we find $X = \{ p_0, p_1, \dots \} \subseteq Y$, such that $(p_i, p_j) \notin V$, for $i < j$. ■

PROPOSITION 8. *For every vicinity U there exists a vicinity W with the following property. If $\phi(x_1, \dots, x_n, \bar{a})$ is an $L(A)$-formula with $\mathrm{MR}[\phi_+, W] < \infty$, there is a largest integer m such that there are types $p_1, \dots, p_m$ satisfying the property $\mathfrak{P}_W(p_1, \dots, p_m)$ defined below.*

$$\mathfrak{P}_W(p_1,\dots,p_m): \quad \begin{cases} p_1, \dots, p_m \in S_n(B), & \text{for some } B \supseteq \bar{a} \\ \phi(\bar{x}, \bar{a}) \in p_i, & \text{for } i = 1, \dots, m \\ \mathrm{MR}[p_i, W] = \mathrm{MR}[\phi_+, W], & \text{for } i = 1, \dots, m \\ (p_i, p_j) \notin U, & \text{for } i < j. \end{cases}$$

PROOF: Take V such that $V \circ V \subseteq U$, and let W correspond to V as in Lemma 6. Let $\mathrm{MR}[\phi_+, W] = \alpha$.

Assume, by way of contradiction, that for each $m < \omega$ there exist $B_m \supseteq \bar{a}$ and $p_1, \dots, p_m \in S_n(B_m)$ such that $\mathfrak{P}_W(p_1, \dots, p_m)$. Use Corollary 4 to extend every type $p \in S_n(B_m)$ with $\mathrm{MR}[p, U] = \alpha$ to a type $q \in S_n(\bigcup_{m<\omega} B_m)$ with $\mathrm{MR}[q, U] = \alpha$.

Let

$$Y = \{ q \in S_n(\bigcup_{m<\omega} B_m) : \phi(\bar{x}, \bar{a}) \in q \text{ and } \mathrm{MR}[q, U] = \alpha \}.$$

By assumption, the set Y is U-large. Applying Lemmas 7 and 6 (in that order), we find $p_0, p_1, \dots$ in Y, and formulas $\psi_0, \psi_1, \dots$ such that $\psi_i \in (p_i)_+$ and the pair $\{\psi_i, \psi_j\}$ is W-separating, for $i < j < \omega$.

Now take $\phi' > \phi$ such that $\mathrm{MR}[\phi', U] = \mathrm{MR}[\phi_+, U] = \alpha$. From the fact that $\phi' \wedge \psi_i \in (p_i)_+$ and $\mathrm{MR}[p_i, U] = \alpha$, we conclude $\mathrm{MR}[\phi' \wedge \psi_i, U] = \alpha$, for each i. Hence $\mathrm{MR}[\phi', U] \geq \alpha + 1$. But this is a contradiction. ■

DEFINITION. The number m given by Proposition 8 is called (U,W)-*Morley degree* of ϕ, and written $\deg[\phi, U, W]$. If Σ is a set of formulas,

$$\deg[\Sigma, U, W] = \min\{ \deg[\sigma, U, W] : \sigma \text{ is a conjunction of formulas in } \Sigma, \text{ and } \mathrm{MR}[\sigma_+, W] = \mathrm{MR}[\Sigma, W] \}$$

We will express the fact that W corresponds to U as in Proposition 8 by saying that "$\deg[-, U, W]$ is defined".

THEOREM 9. *Suppose that $\mathfrak{U}$ is countable. Then the following conditions are equivalent.*

(1) *T is ω-stable with respect to $\mathfrak{U}$.*
(2) $\mathrm{MR}[\phi, U] < \infty$ *for every positive bounded formula ϕ and every vicinity U.*
(3) $\mathrm{MR}[p, U] < \infty$ *for every type p and every vicinity U.*

PROOF: (1)⇒(2) is proved exactly as in classical model theory: Suppose that $\mathrm{MR}[\phi, U] = \infty$. Let

$$\alpha = \sup\{\, \mathrm{MR}[\sigma, U] \,:\, \sigma \text{ is an } L\text{-formula and } \mathrm{MR}[\sigma, U] < \infty \,\}.$$

For any formula σ, we have $\mathrm{MR}[\sigma, U] = \infty$ if and only if $\mathrm{MR}[\sigma, U] > \alpha$. Now, since $\mathrm{MR}[\phi, U] \geq \alpha + 2$, there exist $\psi_1(\bar{x})$ and $\psi_2(\bar{x})$ such that $\{\psi_1(\bar{x}), \psi_2(\bar{x})\}$, is U-separating and $\mathrm{MR}[\phi \wedge \psi_1, U]$, $\mathrm{MR}[\phi \wedge \psi_2, U] \geq \alpha + 1$. But then $\mathrm{MR}[\phi \wedge \psi_2, U] \geq \alpha + 2$ by the choice of α, so we can repeat the argument. Iterating this process ω many times, we obtain an uncountable set of types over a countable set of parameters such any two distinct types in the set are separated by U. Thus, T is not ω-stable.

(2)⇒(3) is trivial.

(3)⇒(1): Take a countable set A, in order to prove that $S(A)$ is separable. For every vicinity U fix $W \subseteq U$ such that $\deg[-, U, W]$ is defined.

For every $L(A)$-formula ϕ, find types

$$q_1^{\phi}, \ldots, q_{\deg[\phi, U, W]}^{\phi}$$

such that $\mathfrak{P}_W(q_1^{\phi}, \ldots, q_{\deg[\phi, U, W]}^{\phi})$, where $\mathfrak{P}_W$ is the property defined in Proposition 8. We can assume that these types are in $S(A)$.

CLAIM. *The set*

$$\bigcup_{\substack{\phi \in L(A) \\ U \in \mathfrak{U}}} \{\, q_1^{\phi}, \ldots, q_{\deg[\phi, U, W]}^{\phi} \,\}$$

is dense in $S(A)$.

Since $L(A)$ and $\mathfrak{U}$ are countable, this will prove that $S(A)$ is separable.

PROOF OF THE CLAIM: Fix $p \in S_n(A)$. Take also $\phi \in p$ be such that $\mathrm{MR}[\phi_+, W] = \mathrm{MR}[p, W]$. By the maximality of $\deg[\phi, U, W]$, there exists $1 \leq i \leq \mathrm{MR}[\phi_+, W]$ such that $(p, q_i^{\phi}) \in U$. ■

5. PRINCIPAL TYPES

If p is a type and U is a vicinity, we denote by $U[p]$ the set $\{\, q \,:\, (p, q) \in U \,\}$. If ϕ is a positive bounded formula, $S_n(A)[\phi]$ denotes the set $\{\, p \in S_n(A) \,:\, \phi \in p \,\}$.

DEFINITION. A type $p \in S_n(A)$ is *principal* with respect to $\mathfrak{U}$ if for every vicinity U there exist an $L(A)$-formula ϕ consistent with $T(A)$ and an approximation ϕ' of ϕ such that

$$S_n(A)[\phi'] \subseteq U[p].$$

The concept of principal type in Banach space model theory was studied in [2].

If ϕ is an $L(\mathfrak{C})$-formula, let $[\,\phi\,] = \{\, p \in S(T) \,:\, \phi \in p \,\}$ The *logical topology* on $S(T)$ is defined as follows. The basic closed sets are the sets of the form $[\,\phi\,]$. This topology is Hausdorff but not compact. (For $m < \omega$, let $\phi_m(x)$ be the formula $\|x\| \geq m$. Then $[\,\phi_0\,] \supseteq [\,\phi_1\,] \supseteq \ldots$, but $\bigcap_{m<\omega}[\,\phi_m\,] = \emptyset$.) It is easy to check that for $p \in S(T)$, the sets $\{\,[\,\psi\,] \,:\, \psi \in p_+\,\}$ form a basic system of neighborhoods of p.

The definition of principality can be reworded as follows. A type p is principal with respect to $\mathfrak{U}$ if, around p, the logical topology is finer than the topology of $\mathfrak{U}$.

REMARKS.

(1) If p and q are types, $d(p,q) < \epsilon$ if and only if

$$\sigma(\bar{x}) \in p \qquad \text{implies} \qquad \exists \bar{z}(\|\bar{z}\| \leq \epsilon \wedge \sigma(\bar{x} + \bar{z})) \in q.$$

Hence, a type $p(\bar{x}) \in S(A)$ is principal with respect to the metric d if and only if for every $\epsilon > 0$ there exist $L(A)$-formulas $\phi(\bar{x}) < \phi'(\bar{x})$, consistent with $T(A)$, such that for every $\sigma(\bar{x}) \in p$,

$$T(A), \phi(\bar{x})' \models_{\mathcal{A}} \exists \bar{z}(\,\|\bar{z}\| \leq \epsilon \wedge \sigma(\bar{x} + \bar{z})\,).$$

(2) A type $p \in S(A)$ is principal with respect to the Banach-Mazur metric if and only if for every $\lambda > 1$ there exist $L(A)$-formulas $\phi < \phi'$, consistent with $T(A)$, such that $T(A), \phi' \models_{\mathcal{A}} p_\lambda$, where p_λ is the set of λ-approximations of formulas in p. (See [3] or [5].)

(3) No type can be principal with respect to the discrete uniform structure: Suppose that $p(\bar{x}) \in S(A)$ is such a type. Then there exist formulas $\phi(\bar{x}) < \phi'(\bar{x})$, consistent with $T(A)$, such that $T(A), \phi' \models_{\mathcal{A}} p$. Take formulas

$$\phi < \psi < \psi' < \phi'.$$

Take also a model E and $\bar{a} \in E$ such that $E \models_{\mathcal{A}} \psi(\bar{a})$. By the perturbation lemma (see [5]), there exists $\delta > 0$ such that $E \models_{\mathcal{A}} \psi'(\bar{b})$, for every $\bar{b} \in E$ such that $\|\bar{a} - \bar{b}\| < \delta$. Thus $E \models_{\mathcal{A}} p(\bar{b})$ for $\|\bar{b} - \bar{a}\| < \delta$. But this is impossible, for every neighborhood of $\bar{a}$ contain elements of different norm, and hence different type.

(4) A type over A which is principal is *almost realized* in every model containing A i.e., if $p \in S(A)$ is principal and $E \supseteq A$ is a model, then for every

vicinity U there exists $q \in S(A)$ such that $(p,q) \in U$ and q is realized in every model of T.

(5) A type over A which is principal with respect to the metric d is *realized* in every model containing A. Let $p \in S(A)$ be principal, and take a model $E \supseteq A$ in order to prove that p is realized in E. First, recall that if $d(q,q') \leq \epsilon$, then for every realization $\bar{c}$ of q there exists a realization $\bar{c}'$ of q' such that $\|\bar{c} - \bar{c}'\| \leq \epsilon$. Using this fact and the preceding remark iteratively, we find a sequence $(\bar{c}_n)_{n\in\omega} \in \mathfrak{C}$ and a sequence $(\bar{d}_n)_{n\in\omega} \in E$ such that for every $n \in \omega$

(i) $\bar{c}_n$ realizes p;
(ii) $\|\bar{c}_n - \bar{d}_n\| \leq 2^{-n}$;
(iii) $\|\bar{d}_n - \bar{c}_{n+1}\| \leq 2^{-n}$.

By (ii) and (iii), the sequences $(\bar{c}_n)$ and $(\bar{d}_n)$ are Cauchy and asymptotically equivalent. Thus, their unique limit is in E. But, by (i) and the perturbation lemma [5], this limit must realize p.

This argument is due to C. W. Henson. In [2], the author proves the converse for A countable. (Omitting Types Theorem for the metric d.)

(6) A type which is principal with respect to the Banach-Mazur metric need not be realized in every model of the theory. Take a sequence

$$1 \leq p_0 < p_1 < \cdots < \infty$$

such that $p_k \to \infty$ as $k \to \infty$. Let

$$E = \bigoplus_{n<\omega} {}_2 \ell_{p_n}(m),$$

where $\ell_p(m)$ denotes the m-dimensional ℓ_p space and $\oplus_2$ is an infinite ℓ_2-sum. Let $p(x,y)$ be the type of a pair of vectors generating 2-dimensional space isometric to $\ell_\infty(2)$. Then p is principal, but is not realized in E.

LEMMA 10. *Let U and V be vicinities such that* $\deg[-,U,V]$ *is defined.*

Suppose $\mathrm{MR}[p(\bar{x}),V] = \alpha$, *and let* $\phi(\bar{x})$ *be a formula in p such that* $\mathrm{MR}[\phi_+,V] = \mathrm{MR}[p,V]$ *and* $\deg[\phi,U,V] = \deg[p,U,V]$. *Then, if* $\{\psi_1,\psi_2\}$ *is a U-separating pair, either* $\psi_1 \notin p$, *or else*

$$\mathrm{MR}[(\phi \wedge \psi_2 \wedge \|\bar{x}\| \leq N)_+, V] < \alpha \qquad \textit{for every } N > 0.$$

PROOF: Suppose that $\psi_1 \in p$ and $\mathrm{MR}[(\phi \wedge \psi_2 \wedge \|\bar{x}\| \leq N)_+, V] \geq \alpha$, in order to reach a contradiction.

Let $m = \deg[p,U,V] = \deg[\phi \wedge \psi_1, U, V]$. Find types $p_1, \ldots p_m \in S_n(A)$ such that

$\phi \wedge \psi_1 \in p_1, \ldots, p_m$;
$\mathrm{MR}[p_i, V] = \alpha$, for $i = 1, \ldots, m$;
$(p_i, p_j) \in U$, for $i \neq j$.

Since $\mathrm{MR}[(\phi \wedge \psi_2 \wedge \|\bar{x}\| \leq N)_+, V] = \alpha$, we can apply Corollary 4 to find a type q such that $\phi \wedge \psi_2 \in q$ and $\mathrm{MR}[q, V] = \alpha$. Since $\{ \psi_1, \psi_2 \}$ is U-separating, $(p_i, q) \notin U$ for $i = 1, \dots, m$. But then have $\deg[\phi, U, V] \geq m + 1$, which is a contradiction. ∎

LEMMA 11. *Suppose that T is ω-stable with respect to $\mathfrak{U}$. Let $\sigma(x_1, \dots, x_n)$ and $\sigma'(x_1, \dots, x_n)$ be consistent $L(A)$-formulas such that $\sigma < \sigma'$. Then there exist $p(\bar{x}) \in S_n(A)$ and formulas $\phi < \phi'$ in p_+ such that*

$$S_n(A)[\phi'] \subseteq S_n(A)[\sigma]' \cap U[p].$$

PROOF: Find a vicinity $V \subseteq U$ such that $\deg[-, U, V]$ is defined.

Let

$$\alpha = \min\{ \mathrm{MR}[p, V] \, : \, p \in S_n(A) \text{ and } \sigma' \in p_+ \}.$$

Take $p \in S_n(A)$ such that $\sigma \in p_+$ and $\mathrm{MR}[p, V] = \alpha$. Choose formulas $\phi < \phi'$ in p_+ such that

$$\mathrm{MR}[\phi', V] = \mathrm{MR}[\phi_+, V] = \mathrm{MR}[p, V] = \alpha,$$
$$\deg[\phi', U, V] = \deg[p, U, V].$$

We may, further, assume that ϕ' is of the form $\sigma' \wedge (\dots)$. Thus$S_n(A)[\phi'] \subseteq S_n(A)[\sigma]'$. We prove that $S_n(A)[\phi'] \subseteq U[p]$. Suppose that this is false. Then there exists $q \in S_n(A)$ such that $\phi' \in q$ and $(p, q) \notin U$. Therefore there is a U-separating pair $\{\psi_1(\bar{x}), \psi_2(\bar{x})\}$ such that $\psi_1 \in p$ $\psi_2 \in q$. By the preceding lemma,

$$\mathrm{MR}[q, U, V] \leq \mathrm{MR}[(\phi \wedge \psi_2 \wedge \|\bar{x}\| \leq \|q\|)_+, V] < \alpha.$$

But this contradicts the minimality of α. ∎

PROPOSITION 12. *Suppose that $\mathfrak{U}$ is countable and T is ω-stable. Then for any consistent $L(A)$-formula $\sigma(x_1, \dots x_n)$ there exists a principal type $p(\bar{x}) \in S_n(A)$ such that $\sigma \in p$.*

PROOF: Take a decreasing sequence of vicinities $U_0 \supseteq U_1 \supseteq \dots$ which is cofinal in $\mathfrak{U}$, i.e., every vicinity contains some U_m. Take also a sequence of formulas $(\sigma_m)_{m<\omega}$ such that $\sigma_0 > \sigma_1 > \dots > \sigma$ and $\lim_{m\to\infty} \sigma_m = \sigma$ (where the limit is taken in the order topology). Using Lemma 11 iteratively, we construct a sequence of types $p_0(\bar{x}), p_1(\bar{x}), \dots$ and sequences of $L(A)$-formulas consistent with $T(A)$

$$\begin{gathered} \phi_0 < \phi_0' \\ \phi_1 < \phi_1' \\ \vdots \end{gathered}$$

such that

$$S_n(A)[\phi'_0] \subseteq S_n(A)[\sigma_0] \cap U_0[p_0],$$
$$S_n(A)[\phi'_{m+1}] \subseteq S_n(A)[\sigma_{m+1} \wedge \phi_m] \cap U_m[p_{m+1}].$$

By the compactness theorem, there exists $p \in S_n(A)$ such that $\phi_m \in p$ for all m. We have $\sigma \in p$ and $(p, p_m) \in U_m$ for all m. It is easy to see that then p must be principal. ∎

6. PRIME MODELS

THEOREM 13. *Suppose that $\mathfrak{U}$ is countable and T is ω-stable with respect to $\mathfrak{U}$. Then for every set A of cardinality κ there exists a first-order model $M \supseteq A$ such that*

(1) *$M = A \cup \{ b_i : i < \gamma \times \kappa \}$, for some $\gamma \leq \omega$;*
(2) *$tp(\, b_j | A \cup \{b_i : i < j\}\,)$ is principal for each $j < \gamma \times \kappa$.*

PROOF: The theorem is proved as in the first-order case, using Lemma 12. ∎

Now we investigate the implications of Theorem 13 for a concrete uniform structure, namely, that of the metric d.

Thanks to the fact that principal types relative to the metric d are realized in every model (see Remark 5 after the definition of principal type), the existence of prime models for theories which are ω-stable with respect to d can be proved, almost literally, as in first-order logic.

THEOREM 14. *Suppose that T is ω-stable with respect to d. Then for every set A, there is a model $E \supseteq A$ with the following property: For any model $F \supseteq A$ there exists an elementary embedding $f : E \prec_{\mathcal{A}} F$, fixing A pointwise.*

If A is separable, the existence of such a model E can be proved from the Omitting Types Theorem for the metric d (see [2]) under an assumption weaker that ω-stability with respect to d, namely, that $S(A)$ is separable with respect to d.

PROOF: Let E be the completion of the model M of Theorem 13. Fix $F \supseteq A$ to define $f : E \prec_{\mathcal{A}} F$. Clearly, it suffices to define f on M.

For $j < \gamma \times \kappa$, we define $f(b_j)$ inductively. Suppose that we have defined $f | A \cup \{ b_i : i < j \}$, in order to define $f(b_j)$. Let $p = tp(\, b_i | A \cup \{ b_i : i < j \})$. Fix $\epsilon > 0$. Since p is principal, there exists $\phi(\bar{x}) \in L(A \cup \{ b_i : i < j \})$ and $\phi' > \phi$ such that for every $\sigma(\bar{x}) \in p$,

$$T(A), \phi'(\bar{x}) \models_{\mathcal{A}} \exists \bar{z}(\, \|\bar{z}\| \leq \epsilon \wedge \sigma(\bar{x} + \bar{z})\,),$$

Hence,

$$T(A), f(\phi'(\bar{x})) \models_{\mathcal{A}} \exists \bar{z}(\, \|\bar{z}\| \leq \epsilon \wedge f(\sigma(\bar{x} + \bar{z}))\,).$$

Since ϵ is arbitrary, we conclude that $f(p)$ is principal. Define $f(b_i)$ as any realization of $f(p)$ in F. ∎

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